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Long-term risk with stochastic interest rates

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Abstract

In constant-rate markets, the average stochastic discount factor growth rate coincides with the instantaneous rate. When interest rates are stochastic, this average growth rate is given by the long-term yield of zero-coupon bonds, which cannot serve as instantaneous discount rate. We show how to reconcile the stochastic discount factor growth with the instantaneous relations between returns and rates in stochastic-rate markets. We factorize no-arbitrage prices and isolate a rate adjustment that captures the short-term variability of rates. The rate-adjusted stochastic discount factor features the same long-term growth as the stochastic discount factor in the market but has no transient component in its Hansen–Scheinkman decomposition, capturing the long-term interest rate risk. Moreover, we show how the rate adjustment can be used for managing the interest rate risk related to fixed-income derivatives and life insurances.

KEYWORDS
forward measure, long-term risk, pricing kernel decomposition, rate adjustment, stochastic interest rates, yields

1 MOTIVATIONS AND MAIN RESULTS

Consider a continuous-time arbitrage-free market in the time interval $[0, T]$, composed of both risky and fixed-income securities that depend on a stochastic instantaneous rate Y_t. Denote by $\pi_t(h_T)$ the no-arbitrage price at time $0 \leqslant t \leqslant T$ of an attainable payoff paid at T. For example, $\pi_t(1_T)$ is the no-arbitrage price of a zero-coupon T-bond (T-ZCB for brevity).

When interest rates are constant over time, the absence of arbitrages leads to the equality

instantaneous expected asset return = instantaneous risk-free rate

under a risk-neutral measure Q. Hansen and Scheinkman (2009) and Hansen (2012) formalize this relation as an eigenvalue-eigenvector problem for the extended infinitesimal generator in a Markov setting. Marinacci and Severino (2018) formulate the same problem in a more general (non-Markov) framework:

$$\mathcal{D}_0 \pi_t = r\, \pi_t, \qquad t \in [0, T]. \tag{1}$$

Here, $\mathcal{D}_0$ is the weak time-derivative, a differential operator that applies to a wide class of semi-martingales and generalizes the infinitesimal generator. This eigenvalue-eigenvector problem captures the absence of arbitrage because Equation (1) is equivalent to $\mathcal{D}_0(e^{-rt}\pi_t) = 0$, that is. discounted no-arbitrage prices are Q-martingales. Importantly, the eigenvalue r has a prominent role in the evolution of the stochastic discount factor since it captures its deterministic growth rate (Hansen & Scheinkman, 2009). Indeed, the pricing kernel in any time interval $[s, t]$ is $M_{s,t} = e^{-r(t-s)} L_{s,t}$, where $L_{s,t}$ is the conditional Radon–Nikodym density of Q with respect to the physical probability.

In a stochastic-rate setting, while Equation (1) may be rephrased at any instant t by using the random rate Y_t instead of r, the sole instantaneous rate Y_t is unable to subsume the stochastic discount factor growth rate on any given time period: the pricing kernel in $[s, t]$ takes the form $M_{s,t} = e^{-\int_s^t Y_\tau d\tau} L_{s,t}$. In this case, defining a growth rate for $M_{s,t}$ is challenging. Under regularity conditions, Qin and Linetsky (2017) individuate a deterministic *long-term yield* as the asymptotic growth rate of the stochastic discount factor, but this quantity is not suitable for characterizing instantaneous returns in the sense of Equation (1).

The main contribution of this paper is to provide a generalization of Equation (1) for stochastic-rate markets, where the employed eigenvalue is determinant for the pricing kernel growth rate. Our generalization is developed in a conditional setting, where we introduce a mathematical instrument to differentiate stochastic processes on the time window $[s, T]$ by disentangling the known information at time s: the *weak time-derivative in* $[s, T]$, denoted by $\mathcal{D}_s$ (Section 2.4). The absence of requirements on the Markov property makes our framework compatible with Qin and Linetsky (2017). See, for example, Qin and Linetsky (2018) for an application in a non-Markov setting.

Our theory is based on the introduction of *rate-adjusted prices* that provide hedging from interest rate variability. The rate-adjusted price of an attainable payoff h_T at time t in the interval $[s, T]$, denoted by $\rho_t^T(s, h_T)$, satisfies

$$\underbrace{\rho_t^T(s, h_T)}_{\text{rate-adjusted price}} = \underbrace{e^{r_s^T (t-s)} \frac{\pi_s(1_T)}{\pi_t(1_T)}}_{\text{adjustment}} \cdot \underbrace{\pi_t(h_T)}_{\text{no-arbitrage price}}, \tag{2}$$

TABLE 1 Summary of results.

	Constant rate	Stochastic rates Finite T	Infinite T
Numéraire	Money market	T-ZCB	Long bond
Measure	Q	F^T	F^∞
Yield	r	r_s^T	r^∞
Price	π	ρ^T	ρ^∞
Return-rate relation	$\mathcal{D}_0\pi = r\pi$	$\mathcal{D}_s\rho^T = r_s^T\rho^T$	$\mathcal{D}_s\rho^\infty = r^\infty\rho^\infty$
Pricing kernel growth rate	r	r_s^T	r^∞

Note: Comparison of numéraires, pricing relations, and pricing kernel growth rates between constant and stochastic rates. Finite and infinite horizons are considered. In the stochastic case, rate-adjusted prices and pricing kernels are considered.

where r_s^T is the yield to maturity of a T-ZCB at time s: $r_s^T = -\log \pi_s(1_T)/(T-s)$. If interest rates are constant, rate-adjusted, and no-arbitrage prices coincide. In general, the two prices are equal at the instants s and T. Interestingly, ρ^T can be interpreted as an indifference price for an investor that ignores the variability of rates, or as the conversion price of a convertible bond (when h_T refers to a stock). In particular, the rate-adjusted price of a ZCB is the no-arbitrage price of the same security if interest rates were constantly equal to r_s^T.

From the standpoint of stochastic discount factors, we define the *rate-adjusted pricing kernel* in the interval $[s,t]$ with $s \leqslant t \leqslant T$ by

$$N_{s,t}^T = e^{-r_s^T(t-s)}\frac{\pi_t(1_T)}{\pi_s(1_T)} M_{s,t}.$$

As expected, $N_{s,t}^T$ and $M_{s,t}$ coincide when interest rates are constant.

Our main results are Theorems 3.2 and 4.3. We prove that rate-adjusted prices and rate-adjusted pricing kernels satisfy the differential relations

$$\mathcal{D}_s\rho_t^T = r_s^T\rho_t^T, \qquad \mathcal{D}_s N_{s,t}^T = -r_s^T N_{s,t}^T \tag{3}$$

for any t in $[s,T]$, where the first equality holds under the forward measure and the second one under the physical measure. These equations parallel the relations satisfied by no-arbitrage prices and by the pricing kernel $M_{s,t}$ when interest rates are constant:

$$\mathcal{D}_s\pi_t = r\pi_t, \qquad \mathcal{D}_s M_{s,t} = -rM_{s,t},$$

where the two equalities hold under the risk-neutral measure Q and the physical measure, respectively. The equations on the left rephrase the relation between returns and rates, while the parameters in the right-hand side equalities identify the pricing kernel growth rates.

Results similar to Equation (3) are unlikely to hold for no-arbitrage prices when rates are stochastic: the instantaneous rate Y_t cannot serve as eigenvalue and the transitory component of $M_{s,t}$ hinders the differentiation. Nonetheless, rate-adjusted prices are able to reach the purpose. ZCB yields play both the role of eigenvalues (for rate-adjusted prices) and that of growth rates (for rate-adjusted pricing kernels), generalizing the features of the constant-rate case. See Table 1.

The results are robust when the horizon T is moved to infinity (Subsections 3.5 and 4.2): the long-term yield arises as the growth rate for both the rate-adjusted pricing kernel and $M_{s,t}$. Indeed, the long-term rate-adjusted pricing kernel $N_{s,t}^\infty$ differs from $M_{s,t}$ only in the transient component in the Hansen and Scheinkman (2009) decomposition. Such component is trivial only for $N^{\infty}_{s,t}$ when interest rates are stochastic. So, pricing with the rate adjustment means overlooking the transient component in the pricing kernel and, hence, focusing on the long-term interest rate risk.

Precisely, when T goes to infinity, the adjustment in Equation (2) converges to the transient component of $M_{s,t}$. The adjustment can, then, be associated with temporary rate fluctuations and Equation (2) is actually a decomposition of the no-arbitrage price into a rate-adjusted price (sensible to long-lasting shocks in the yield curve) and the adjustment (sensible to temporary rate variations). We are, then, providing a way to disentangle the short- and long-term interest rate risks of a given cashflow. Therefore, our theory has promising applications to the risk management of financial and insurance products with distant maturities and significant exposure to interest rate variability, as fixed-income derivatives, life insurances and annuities (Section 5). Section 6 illustrates the theory in the context of single-factor affine interest rate models. The appendix provides several theoretical issues and the proofs.

2 | ASSET PRICING FRAMEWORK

We start with describing a general continuous-time arbitrage-free market with stochastic interest rates. We then discuss the conditions that ensure long-term convergences, the forward measures and the properties of weak time-derivatives. All technical details are in Appendix A.

2.1 | Arbitrage-free market

We fix the time horizon $T > 0$ and consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where the filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ satisfies the usual conditions and is left-continuous in T. We call P the *physical measure*. An adapted random process $Y = \{Y_t\}_{t\in[0,T]}$ represents the instantaneous interest rate and $\{e^{\int_0^t Y_\tau d\tau}\}_{t\in[0,T]}$ is the money market account. ZCBs with any possible maturity (and face value of one unit) are traded. Additional risky securities, with adapted price processes, can be present in the market, too.

If $X = \{X_t\}_{t\in[0,T]}$ denotes the price process of a traded security, at any instant t its relative price $\tilde{X}_t = e^{-\int_0^t Y_\tau d\tau} X_t$ is obtained by discounting with the money market account. We assume that all relative asset prices are semimartingales and there exists an equivalent martingale measure Q (the *risk-neutral measure*). Hence, arbitrage opportunities are ruled out. We indicate by L_T the Radon–Nikodym derivative of Q with respect to P on $\mathcal{F}_T$, that is, $L_T = dQ/dP$, and we set $L_t = \mathbb{E}_t[L_T]$ for all $t \in [0, T]$, where $\mathbb{E}_t$ is the conditional expectation with respect to $\mathcal{F}_t$ under P. We also define $L_{t,T} = L_T/L_t$.

We denote by $M = \{M_t\}_{t\in[0,T]}$ the strictly positive stochastic discount factor process associated with Q, namely $M_t = e^{-\int_0^t Y_\tau d\tau} L_t$ for all t. In addition, we define the pricing kernel in any time interval $[s, t]$ with $s \leqslant t \leqslant T$ by $M_{s,t} = M_t/M_s = e^{-\int_s^t Y_\tau d\tau} L_{s,t}$.

The no-arbitrage price at time t of a T-ZCB is $\pi_t(1_T)$ and the related *yield to maturity* is $r_t^T = -\log \pi_t(1_T)/(T - t)$. We also define r_T^T as the a.s. limit of r_t^T when t approaches T. Clearly, if interest rates are constant over time (and deterministic), the yield coincides with the instantaneous rate.

The time horizon T is fixed in several sections of the paper while in other parts, we move the horizon T to infinity (Subsections 2.2, 2.3, 3.4, 3.5, 4.1, 4.2, 5.2, and 6.1.2). When T increases, any payoff h_T paid at time T also moves farther in time and, in particular, the T-ZCB features a farther maturity. This case is considered, for example, in Subsections 3.4 and 6.1.2. However, in Subsection 3.5, the payment date is disentangled from the time horizon T and only the latter moves to infinity. This approach permits to illustrate, in Subsection 3.6, rate-adjusted pricing for a given cashflow where only the last payment may coincide with the time horizon T.

2.2 | Long-term assumptions

Our semimartingale framework is compatible with the setting of Qin and Linetsky (2017) that formalize the convergence of bond yields, forward measures and stochastic discount factors when the horizon T becomes larger and larger.

Assumptions 2.1. We assume the following:

- (i) M_t is a strictly positive semimartingale such that $\mathbb{E}[M_{t,T}]$ is finite for all $0 \leqslant t < T$.
- (ii) For all $t > 0$, when T goes to infinity, $\mathbb{E}_t[M_T]/\mathbb{E}[M_T]$ converges in L^1 to a positive $\mathcal{F}_t$-measurable random variable G_t^∞.
- (iii) For all $t > 0$, when T goes to infinity, $\pi_0(1_{T-t})/\pi_0(1_T)$ has a positive finite limit.
- (iv) For all $t > 0$, when T goes to infinity, the limit in probability of r_t^T is positive.

The three first assumptions are from Qin and Linetsky (2017) and ensure that, for any t, the yield to maturity r_t^T converges in probability to a deterministic *long-term yield* r^∞ (Theorem 3.2 therein). This result is consistent with the persistence of the yield curve over large maturities (Diebold & Li, 2006). Notably, r^∞ is not dependent on t, consistently with the impossibility of falling long-term rates (Dybvig et al., 1996). The fourth assumption additionally ensures that r^∞ is positive. Interestingly, Gouriéroux et al. (2023) provide a different set of assumptions in order to deal with stochastic long forward rates.

The long-term yield is associated with a long bond, obtained by a roll-over portfolio of ZCBs across increasing maturities. The time t value of this portfolio is denoted by B_t^∞, while b_t^∞ is its long-term discounted value: $b_t^\infty = e^{-r^\infty t} B_t^\infty$.

2.3 | Forward measures and pricing

By using as numéraire the no-arbitrage price of a T-ZCB, we construct the forward measure with horizon T or, simply, T-*forward measure* and we denote it by F^T (Geman et al., 1995). This probability measure is equivalent to Q and we indicate its Radon–Nikodym derivative with respect to P on $\mathcal{F}_T$ by $G_T^T = dF^T/dP$. Moreover, we set $G_t^T = \mathbb{E}_t[G_T^T]$ for any $t \in [0, T]$ and we define $G_{t,T}^T = G_T^T / G_t^T$. In case, interest rates are constant, F^T coincides with Q. Using G_t^T, the stochastic discount factor and the pricing kernel in any interval $[s, t]$ may be expressed as

$$M_t = e^{r_t^T (T-t) - r_0^T T} G_t^T, \qquad M_{s,t} = e^{r_t^T (T-t) - r_s^T (T-s)} G_{s,t}^T.$$

Although $M_{s,t}$ refers to the time window $[s, t]$, its expression depends on the horizon T through the density of the T-forward measure.

The T-forward measure provides a handy representation of no-arbitrage prices. Consider, for instance, an attainable $\mathcal{F}_T$-measurable payoff h_T such that $\mathbb{E}_s^{F^T}[|h_T|]$ is $\mathcal{F}_s$-measurable. Its no-arbitrage price at any time $t \in [s,T]$ can be written in equivalent ways, depending on the numéraire:

$$\pi_t(h_T) = \mathbb{E}_t^Q\left[e^{-\int_t^T Y_\tau d\tau} h_T\right] = e^{-r_t^T(T-t)}\mathbb{E}_t^{F^T}[h_T]. \tag{4}$$

The right-hand side makes a fruitful bridge between constant-rate and stochastic-rate valuation.

We now consider the case in which the horizon T becomes arbitrarily large under the Assumptions 2.1. Qin and Linetsky (2017) prove in Theorem 3.1 that, on each $\mathcal{F}_t$, F^T strongly converges to the *long-term forward measure* F^∞ when T goes to infinity. At any time t, we denote the Radon–Nikodym derivative of F^∞ with respect to P by $G_t^\infty = dF^\infty/dP|_{\mathcal{F}_t}$ and the collection of all G_t^∞ constitutes a P-martingale. In addition, we set $G_{s,t}^\infty = G_t^\infty/G_s^\infty$ for all s and t. Interestingly, the long-term forward measure is related to a specific numéraire: the price B_t^∞ of the long bond.

Finally, Qin and Linetsky (2017) prove in Theorem 3.2 that the pricing kernel $M_{s,t}$ satisfies the long-run decomposition

$$M_{s,t} = e^{-r^\infty(t-s)}\,\frac{b_s^\infty}{b_t^\infty}\,G_{s,t}^\infty. \tag{5}$$

In a Markov environment, $G_{s,t}^\infty$ defines the *permanent* (or *martingale*) *component* of Hansen and Scheinkman (2009) decomposition, while b_s^∞/b_t^∞ constitutes the *transient component* and r^∞ is the deterministic *long-term growth rate*. For the empirical estimation of these terms, see Christensen (2017) and Qin et al. (2018). Furthermore, from Equation (5), it is easy to write the no-arbitrage price of any payoff h_T by using the long-term forward measure:

$$\pi_t(h_T) = e^{-r^\infty(T-t)}\mathbb{E}_t^{F^\infty}\left[\frac{b_t^\infty}{b_T^\infty}h_T\right]. \tag{6}$$

2.4 | Weak time-derivative in $[s,T]$

To analyze the increments of stochastic processes, we need a differential operator. Marinacci and Severino (2018) introduce the *weak time-derivative* that applies to a wide class of semimartingales. Here we extend this notion to a conditional setting. Details and proofs are in Appendix A.

We fix an instant $s \in [0,T]$ and we consider the conditional space $L_s^1(\mathcal{F}_T)$ composed of variables $f \in L^0(\mathcal{F}_T)$ such that $\mathbb{E}_s[|f|] \in L^0(\mathcal{F}_s)$ and endowed with the L^0-valued metric $d(f,g) = \mathbb{E}_s[|f - g|]$. Moreover, we denote by $\mathcal{U}_s$ the L^0-module of adapted processes $u : [s,T] \to L_s^1(\mathcal{F}_T)$ that are L_s^1-right-continuous in $[s,T)$ and L_s^1-left-continuous at T. We then define the weak time-derivative for processes in $\mathcal{U}_s$. For any $t \in [s,T]$, the definition uses the space $C_c^1((t,T), L^0(\mathcal{F}_s))$ of functions $\varphi_s : [t,T] \to L^0(\mathcal{F}_s)$ that have compact support in (t,T) and are continuously differentiable over time. The integrals in the next definition are pathwise integrals of processes in $L^0(\mathcal{F}_s)$.

Definition 2.2. We say that a process $u \in \mathcal{U}_s$ is *weakly time-differentiable in* $[s,T]$ when there exists a process $v \in \mathcal{U}_s$ such that for every $t \in [s,T]$

$$\int_t^T \mathbb{E}_s[v_\tau \mathbf{1}_{A_t}]\varphi_s(\tau)d\tau = -\int_t^T \mathbb{E}_s[u_\tau \mathbf{1}_{A_t}]\varphi_s'(\tau)d\tau$$

for all $A_t \in \mathcal{F}_t$ and $\varphi_s \in C_c^1((t,T), L^0(\mathcal{F}_s))$. We call v a *weak time-derivative* of u in $[s,T]$.

Definition 2.2 generalizes the weak time-derivative of Definition 2.1 in Marinacci and Severino (2018), where $s = 0$ and deterministic test functions are employed. The weak time-derivative in $[s,T]$ is unique (Proposition A.2 in Appendix A) and we denote it by $\mathcal{D}_s u$. Then, we introduce the L^0-submodules of $\mathcal{U}_s$, denoted by $\mathcal{U}_s^1$ and $\mathcal{U}_s^\infty$, that consist of weakly (or infinitely weakly) time-differentiable processes in $[s,T]$.

As shown in Marinacci and Severino (2018), the notion of weak time-derivative applies to any adapted process (with few regularity conditions), differently from the infinitesimal generator (requiring the Feller property) and the extended infinitesimal generator (requiring the Markov property). However, when these properties hold, we retrieve the infinitesimal generator and its extended version. Compared to these alternatives, the advantage of using weak time-derivatives consists in the larger class of processes to consider, which ensures a greater generality of the results.

Example 2.3. In Heath et al. (1992), the bond price process follows the dynamics

$$d\pi_t(1_T) = \left(Y_t + b_t^T\right)\pi_t(1_T)dt + \sum_{i=1}^n a_{i,t}^T \pi_t(1_T) dW_{i,t}^P,$$

where W_t^P is an n-dimensional independent P-Wiener process and b^T and a_i^T are adapted processes. Here, $\pi_t(1_T)$ is non-Markovian because the drift and the diffusion coefficients may depend on the whole history. See Equation (9) in Heath et al. (1992), the notation and the technical assumptions therein. In this case, the no-arbitrage condition on the bond price can be properly stated in terms of weak time-derivatives:

$$\mathcal{D}_0 \pi_t(1_T) = Y_t \pi_t(1_T) = \left(Y_t + b_t^T - \sum_{i=1}^n a_{i,t}^T \nu_{i,t}\right)\pi_t(1_T),$$

where ν_t is the market price of risk vector and $dW_t^Q = dW_t^P + \nu_t dt$ is a Q-Wiener process, according to Girsanov's theorem. This step directly leads to Condition C.4 in Heath et al. (1992), that is,

$$b_t^T - \sum_{i=1}^n a_{i,t}^T \nu_{i,t} = 0.$$

As to weak time-derivative in $[s,T]$, it is important to notice that $\mathcal{F}_s$-measurable functions play the role of multiplicative constants. Given a process $u \in \mathcal{U}_s^1$ and $\xi_s \in L^0(\mathcal{F}_s)$, the process defined by $\xi_s u_t$ for all $t \in [s,T]$ belongs to $\mathcal{U}_s^1$, too, and $\mathcal{D}_s(\xi_s u) = \xi_s \mathcal{D}_s u$. This property permits to deal with $\mathcal{F}_s$-measurable parameters in the differential equations and to study eigenvalue-eigenvector problems for $\mathcal{D}_s$ with $\mathcal{F}_s$-measurable eigenvalues.

The key feature of weak time-derivatives in $[s,T]$ is the characterization of *conditional* (or *generalized*) *martingales*. By this terminology, we mean processes u defined in the time interval $[s,T]$ with all the properties of martingales except for integrability, which is replaced by the weaker condition $\mathbb{E}_t[u_\tau] \in L^0(\mathcal{F}_t)$ for all $s \leqslant t \leqslant \tau \leqslant T$ (Shiryaev, 1996, Chapter VII, §1). Importantly, a process belongs to $\mathcal{U}_s^1$ and has null weak time-derivative in $[s,T]$ if and only if it is a conditional martingale (Proposition A.4 in Appendix A). For a use of conditional martingales in portfolio optimization, see Cerreia-Vioglio et al. (2022).

When the risk-neutral probability Q is considered, an example of conditional martingale is provided by the process of *futures prices* of a claim with expiry T at any t in $[s, T]$. On the contrary, under the forward measure, *forward prices* in $[s, T]$ for the settlement date T are conditional martingales (Musiela & Rutkowski, 2005, Sections 9.6 and 11.5). Both these processes exhibit null weak time-derivatives in $[s, T]$ with respect to different measures. The weak time-derivative in $[s, T]$ is also useful for the analysis of no-arbitrage prices because, after discounting by the money-market account, they are martingales under Q. Moreover, the weak time-derivative captures the drift of semimartingale processes and applies to a wide range of continuous-time pricing models. See Marinacci and Severino (2018) and the example in Subsection 3.1.

3 | PRICING EQUATION

We formulate and solve a *rate-adjusted* pricing equation for the valuation of random payments in a market with stochastic interest rates. We, then, compare the solution of the equation with the usual risk-neutral pricing formula. We interpret rate-adjusted prices as indifference prices and as conversion prices for hybrid securities. We finally study their properties in the long run and we describe the valuation of cashflows. Proofs can be found in Appendix B.

3.1 | Rate-adjusted and no-arbitrage pricing

In the time interval $[s, T]$, we face the problem of evaluating an $\mathcal{F}_T$-measurable payoff h_T. As illustrated in Subsection 2.3, we can express the time t no-arbitrage price of h_T by using both the measure Q and the forward measure. Up to replacing the risk-neutral measure with F^T and instantaneous rates with bond yields, the right-hand side of Equation (4) formally traces no-arbitrage pricing with constant interest rates.

Generalizing Hansen and Scheinkman (2009), Marinacci and Severino (2018) show that, in an arbitrage-free market with constant interest rate r, asset prices satisfy the relation (1). In particular, the risk-neutral price process $\pi_t(h_T)$ is the unique solution in $\mathcal{U}_0^1$ of the no-arbitrage pricing equation

$$\begin{cases} \mathcal{D}_0 f_t = r\, f_t & t \in [0, T) \\ f_T = h_T \end{cases} \tag{7}$$

where $h_T \in L^1(\mathcal{F}_T, Q)$. This eigenvalue-eigenvector problem has its roots in the Perron–Frobenius theory and captures the essence of no-arbitrage. Indeed, it relates infinitesimal price increments of a possibly risky security with the interest rate deriving from a locally riskless investment. This approach is known in the literature since Cox and Ross (1976). Moreover, the rate r, up to a sign change, defines the growth rate of the stochastic discount factor in the Hansen–Scheinkman decomposition, regardless of the time horizon.

Example 3.1. Consider a Black and Scholes (1973) market with a riskless bond and a risky security whose prices follow the dynamics (under the physical measure)

$$dB_t = rB_t dt, \qquad dX_t = \mu X_t dt + \sigma X_t dW_t^P$$

with $r, \mu \in \mathbb{R}, \sigma > 0$, and W_t^P a standard Wiener process. It is apparent that the bond price satisfies problem (7). As for the risky asset, under P, the Wiener process W_t^P exhibits null weak time-derivative because it is a martingale, but the drift of X_t differs from rX_t. However, moving to the risk-neutral measure Q, the dynamics of X_t becomes

$$dX_t = rX_t dt + \sigma X_t dW_t^Q, \tag{8}$$

where W_t^Q is a Q-Wiener process, and so $\mathcal{D}_0 X_t = rX_t$. Hence, problem (7) is satisfied. The risk-free and the risky security share the same drift coefficient under Q, given by the instantaneous rate.

The dynamics presented so far generalize to stochastic-rate settings by replacing r with the instantaneous rate Y_t. For example, Chapter 1 of Karatzas and Shreve (1998) derives similar dynamics to Equation (8) with random rates. Nevertheless, there are some flaws in this kind of generalizations. First, the drift coefficient Y_t is unable to capture the stochastic discount factor growth rate over any time period, differently from the constant-rate case. Second, Y_t is a random process, it is not known ex ante, and it forbids an eigenvalue-eigenvector formulation of the problem in the spirit of Hansen and Scheinkman (2009).

Our generalization of problem (7) solves these issues for any given dynamics of interest rates. We employ the forward measure instead of Q and T-bond yields in place of short-term rates to formulate a suitable eigenvalue-eigenvector problem. Moreover, we provide a conditional version of the differential problem defined on any time window $[s, T]$ by using the weak time-derivative in $[s, T]$. The arising eigenvalue is an $\mathcal{F}_s$-measurable random variable (known at the beginning of the trading interval) and, consistently, the growth rate of the stochastic discount features the same $\mathcal{F}_s$-measurability.

We now place ourselves in the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, F^T)$, where we employ the forward measure. We solve in $\mathcal{U}_s^1$ the following pricing differential equation with random coefficient given by the yield to maturity $r_s^T \in L^0(\mathcal{F}_s)$, that is,

$$\begin{cases} \mathcal{D}_s f_t = r_s^T f_t & t \in [s, T) \\ f_T = h_T \end{cases} \tag{9}$$

with $h_T \in L_s^1(\mathcal{F}_T, F^T)$. We refer to Equation (9) as the *rate-adjusted pricing equation*.

Theorem 3.2. *There exists a unique solution of problem (9) in $\mathcal{U}_s^1$, given by*

$$\rho_t^T(s, h_T) = e^{-r_s^T (T-t)} \mathbb{E}_t^{F^T}[h_T] \qquad \forall t \in [s, T]. \tag{10}$$

We refer to $\rho_t^T(s, h_T)$ as the *rate-adjusted price* of h_T at time t in the interval $[s, T]$.

Theorem 3.2 is substantially an assessment of the conditional martingale property of the (forward price) process $\{e^{r_s^T (T-t)} \rho_t^T\}_{t \in [s,T]}$ under F^T. Under specific assumptions on asset and rate dynamics, the problem can be tackled from the perspective of the Feynman–Kaç PDE (see Subsection 6.2.1).

At any instant t in $[s, T]$, r_s^T is the only average rate employed by ρ_t^T for the valuation on h_T. The valuation instant is synchronous with the information set only for the initial ρ_s^T. In particular, ρ_s^T coincides with the no-arbitrage price π_s. The two coincide also at the terminal date. When $s < t < T$, the rate-adjusted price is different from the no-arbitrage price. Fixed any t, a bunch of valuations $\rho_t^T(s, h_T)$ are available, obtained by solving several problems as Equation (9) defined on different time intervals $[s, T]$ with $s < t$. However, rate-adjusted prices with different starting points are consistent within them. If $s_1 \leqslant s_2 \leqslant t$ and $h_T \in L^1_{s_1}(\mathcal{F}_T, F^T)$, the martingale property of forward prices ensures

$$\mathbb{E}_{s_1}^{F^T}\left[e^{r_{s_2}^T(T-t)}\rho_t^T(s_2, h_T)\right] = \mathbb{E}_{s_1}^{F^T}\left[e^{r_{s_1}^T(T-t)}\rho_t^T(s_1, h_T)\right].$$

In addition, as $s_2 \to t^-$, if $r_{s_2}^T$ converges in probability to r_t^T, by the continuous mapping theorem, we get $\rho_t^T(s_2, h_T) \xrightarrow{P} \pi_t(h_T)$. The proper link between ρ^T and π is given by

$$\rho_t^T(s, h_T) = e^{-(r_s^T - r_t^T)(T-t)}\, \pi_t(h_T) = e^{r_s^T(t-s)} \frac{\pi_s(1_T)}{\pi_t(1_T)}\, \pi_t(h_T). \tag{11}$$

This equality can also be read as a parity relation between ρ^T and π:

$$e^{r_s^T(T-t)}\, \rho_t^T(s, h_T) = e^{r_t^T(T-t)}\, \pi_t(h_T),$$

where both sides deliver the price at time t of a forward contract on h_T for date T.

The difference between rate-adjusted and no-arbitrage prices is genuinely due to the term structure of interest rates: in case rates are constant, the distortion between ρ^T and π disappears. In problem (9), ρ^T solves in $[s, T]$ the analogous differential equation for π in problem (7), where interest rates are replaced by $\mathcal{F}_s$-measurable bond yields and the forward measure replaces the risk-neutral one. In this perspective, rate-adjusted prices may be seen as a generalization of no-arbitrage prices in floating-rate markets. Moreover, when ZCBs are considered, $\rho_t^T(s, 1_T)$ is exactly the no-arbitrage price of the ZCB if interest rates were constantly equal to the yield over $[s, T]$.

Figure 1 considers U.S. Treasury bonds in January 1993 with increasing expiry T of 5, 10, 20, and 30 years. We fix $s = 0$ at January 1993 and consider annual t up to the redemption date. Data are provided by the Federal Reserve Board at daily frequency (Gürkaynak et al., 2007). In particular, the yield curve associated with U.S. Treasury bonds in January 1993 is increasing. In the four graphs of Figure 1, we plot the rate-adjusted prices $\rho_t^T(0, 1_T)$ of these securities and the ex post realizations of no-arbitrage prices $\pi_t(1_T)$, observed in later years in the market. Treasury bonds rate-adjusted and no-arbitrage prices are indistinguishable at the initial date and near maturity. Different values of r_0^T may, however, induce an overestimation or underestimation of bond prices: $\rho_t^T(0, 1_T) < \pi_t(1_T)$ if and only if $r_0^T > r_t^T$, a property that holds for any non-negative payoff h_T, too.

3.2 | Rate-adjusted prices as indifference prices

Rate-adjusted prices can be interpreted as indifference prices that allow investors to hedge from interest rates variability.

We consider an investor who incurs an expenditure of $\pi_s(1_T)$ at time s in order to buy a self-financing portfolio that delivers some units of a risky security with payoff h_T at maturity. Specifically, at date s, she buys a T-ZCB. At a later time t, she rebalances her allocation, worth $\pi_t(1_T)$, by purchasing the amount $\pi_t(1_T)/\pi_t(h_T)$ of the risky security. Then, she holds this portfolio until maturity.

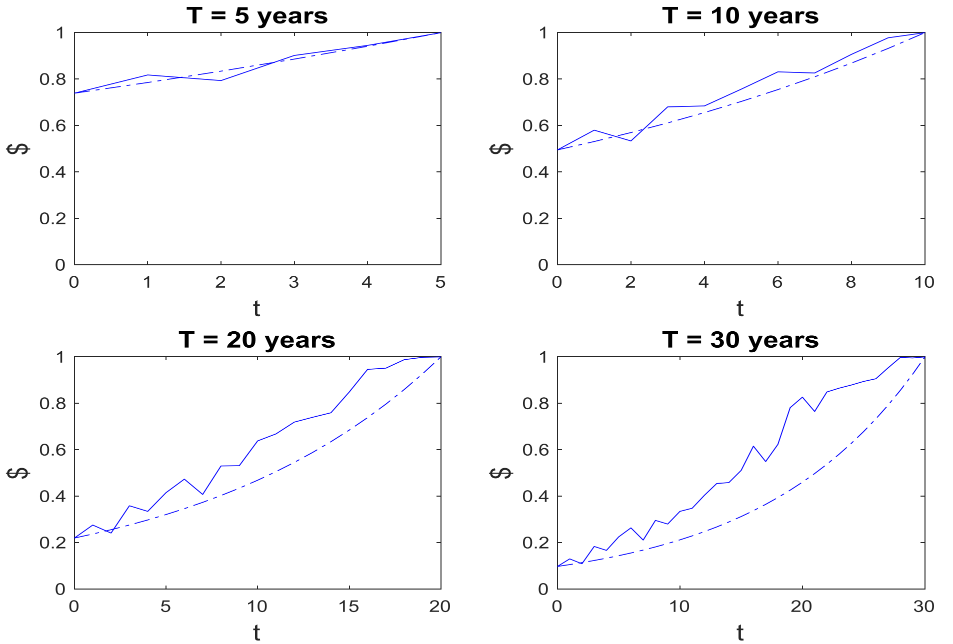


FIGURE 1 Fix $s = 0$ at January 1993 and consider increasing maturities T of U.S. Treasury bonds. Dashed lines depict the annual values of $\rho_t^T(0, 1_T)$ in $[0, T]$. Solid lines represent the (ex post) realizations of no-arbitrage prices $\pi_t(1_T)$ at any year t until expiration. [Color figure can be viewed at wileyonlinelibrary.com]

We suppose that another investor faces the same market. She is less sophisticated than the previous one and pretends that the ZCB is traded at the fixed rate r_s^T. Her belief is consistent with the observation of the T-bond price at time s, because $\pi_s(1_T) = e^{-r_s^T(T-s)} = \rho_s^T(s, 1_T)$. Similarly to the first agent, she plans to go long on the ZCB at time s and to entirely liquidate her position at t in order to purchase all the units of the risky security that she can afford.

Since the second investor disregards the term structure of bond prices, she expects a position worth $e^{-r_s^T(T-t)}$ at time t and, consequently, an erroneous amount of the risky security at maturity. However, the two agents would end up with the same number of units of h_T if the second investor used ρ^T for pricing the risky security. Indeed, $\rho_t^T(s, h_T)$ is the theoretical price of h_T that makes the terminal values of both portfolios coincide.

This description substantiates the hedging nature of ρ^T. Although the first investor properly exploits the variability of rates, the second one assumes a flat term structure. Given an identical initial expenditure, they obtain the same outcome if the less sophisticated agent employs rate-adjusted prices for the valuation of marketed payoffs.

The interpretation provided here is compatible with the rational inattention paradigm. In rational inattention models, individuals have limited capacity for processing information and optimally choose which pieces of information to focus on (see, e.g., Sims (2011)). Agents can choose the frequency to inform themselves, as well as the accuracy and the object of the information, depending on the cost of information processing. In particular, inattentiveness can be about the term structure of interest rates. As Zhang et al. (2020) remark, yields of illiquid Treasuries are rarely observed, and the term structure of interest rates can be deduced only by means of numerical methods. As a result, it is costly and time-consuming to exactly acquire the interest rates. According to Reis (2006), the level of inattentiveness is also related to the magnitude of interest rates: high interest rates induce more timely revisions of consumption-saving choices. Moreover, in Maćkowiak and Wiederholt (2015), households choose to pay little attention to real interest rates and, in Civelli et al. (2019), rationally inattentive agents cannot fully observe the interest rates set by the monetary policy authority. Hence, our interpretation of rate-adjusted prices can be recast in terms of attentive versus inattentive investors about the evolution of interest rates.

3.3 | Rate-adjusted prices as conversion prices

We now provide an interpretation of rate-adjusted prices in terms of conversion prices of mandatory convertible bonds with contingent conversion prices. We assume that the payoff h_T represents a stock price and we denote it by X_T.

We consider a convertible bond issued at time s with maturity T, unitary price and fixed rate r_s^T. At a given time $t < T$, the security is compulsorily converted into a proper amount of shares. The conversion price and ratio are determined at time t by no-arbitrage considerations.

Suppose that at time s, the issuer of the hybrid security purchases $1/\pi_s(1_T)$ units of a T-ZCB. The total cashflow at s is null. Immediately before the conversion, the value of the position is $\pi_t(1_T)/\pi_s(1_T)$ while the convertible bond is worth $e^{r_s^T(t-s)}$. On the one hand, the holdings of the issuer at time t correspond to $\pi_t(1_T)/(\pi_s(1_T)\pi_t(X_T))$ shares with price $\pi_t(X_T)$. On the other, the bond conversion takes place according to the relation $e^{r_s^T(t-s)} = p_t q_t$, where p_t and q_t are $\mathcal{F}_t$-measurable conversion price and ratio. The absence of arbitrage opportunities implies that q_t needs to coincide with the amount of shares $\pi_t(1_T)/(\pi_s(1_T)\pi_t(X_T))$. Otherwise, an initial null expenditure would ensure a positive outcome (in terms of stocks) at terminal date T. Since $p_t = e^{r_s^T(t-s)}/q_t$, we deduce that p_t takes the expression of Equation (11): p_t equals the rate-adjusted price $\rho_t^T(s, X_T)$.

The construction of the previous hybrid asset characterizes ρ^T as a conversion price from bonds to stocks in an arbitrage-free market. The reasoning highlights the ability of ρ^T to quantify the risk exposure when financing through fixed-rate securities in a market with stochastic rates.

3.4 | Long-term relations between rate-adjusted and no-arbitrage prices

We now investigate the relation between ρ_t^T and π_t when the horizon T increases.

Proposition 3.3. *Under the Assumptions 2.1, if $h_T \in L_s^1(\mathcal{F}_T, F^T)$, then for any $t > s$*

$$\frac{\rho_t^T(s, h_T) - \pi_t(h_T)}{\mathbb{E}_t^{F^T}[h_T]} \xrightarrow{P} 0, \qquad T \to +\infty$$

and, in case h_T is also strictly positive,

$$\frac{\log \rho_t^T(s, h_T) - \log \pi_t(h_T)}{T - t} \xrightarrow{P} 0, \qquad T \to +\infty.$$

The asymptotic behaviors of Proposition 3.3 are valid for any choice of s and $t > s$, which are fixed before taking the limit across increasing horizons. However, convergences are taken after a proper rescaling. We now focus on the ratio between rate-adjusted and no-arbitrage prices.

Proposition 3.4. *Under the Assumptions 2.1, if $h_T \in L^1_s(\mathcal{F}_T, F^T)$, then for all $t > s$*

$$\mathbb{E}_s^{F^T}\left[\frac{\rho_t^T(s,h_T)}{\pi_t(h_T)}\right] \xrightarrow{P} e^{(r^\infty - r_s^t)(t-s)}, \qquad T \to +\infty$$

and

$$\frac{\rho_t^T(s,h_T)}{\pi_t(h_T)} \xrightarrow{P} \frac{b_s^\infty}{b_t^\infty}, \qquad T \to +\infty.$$

The difference between the long-term yield and the bond yield r_s^t determines the limit for the expected ratio between ρ_t^T and π_t under F^T. In addition, this ratio turns out to have a well-defined long-run limit in probability. Asymptotically, ρ_t^T differs from π_t by a multiplicative $\mathcal{F}_t$-measurable factor associated with the discounted long bond. This factor is actually the transient component in the long-term pricing kernel decomposition of Equation (5).

3.5 | Rate-adjusted prices as long-term prices

As described by Qin and Linetsky (2017) and recapped in Equation (5), under the Assumptions 2.1, the long-term growth rate of the pricing kernel $M_{s,t}$ is the long-term yield r^∞. Since r^∞ is the limit of r_s^T when T goes to infinity and r_s^T is the leading parameter of problem (9), we move T to infinity in this problem and analyze the solutions over increasing horizons. We begin with considering the *long-term* rate-adjusted problem

$$\begin{cases} \mathcal{D}_s f_t = r^\infty\, f_t & t \in [s,T) \\ f_T = h_T \end{cases} \tag{12}$$

with $h_T \in L^1_s(\mathcal{F}_T, F^\infty)$. Differently from problem (9), here the long-term yield replaces r_s^T and the long-term forward measure is employed. Theorem 3.2 ensures that problem (12) has a unique solution in $\mathcal{U}_s^1$, given by

$$\rho_t^\infty(s,h_T) = e^{-r^\infty(T-t)}\mathbb{E}_t^{F^\infty}[h_T] \qquad \forall t \in [s,T].$$

We refer to $\rho_t^\infty(s,h_T)$ as the *long-term rate-adjusted price* of h_T at time t in the interval $[s,T]$.

To investigate the convergence of ρ^T to ρ^∞ (when the horizon increases), we solve a sequence of differential problems related to a term structure of horizons, whose solutions tend to the solution of problem (12). To do so, we disentangle the instant at which the payoff under scrutiny is paid (τ) from the horizon ($T \geqslant \tau$). The rate-adjusted pricing problem of Subsection 3.1 can, then, be rewritten as

$$\begin{cases} \mathcal{D}_s f_t = r_s^T\, f_t & t \in [s,\tau) \\ f_\tau = h_\tau \end{cases} \tag{13}$$

with $h_\tau \in L^1_s(\mathcal{F}_\tau, F^T)$. If $\tau = T$, we retrieve problem (9). In the more general formulation considered here, the unique solution in $\mathcal{U}^1_s$ over $[s,\tau]$ is

$$\rho^T_t(s, h_\tau) = e^{-r^T_s(\tau-t)}\mathbb{E}^{F^T}_t[h_\tau] \qquad \forall t \in [s,\tau]. \tag{14}$$

This rate-adjusted price is fundamental for the valuation of cashflows in the next subsection. In addition, under mild assumptions, it converges to the long-term rate-adjusted price.

Proposition 3.5. *Under the Assumptions 2.1, suppose that $h_\tau \in L^1(\mathcal{F}_\tau, F^T)$ for all $T \geqslant \tau$ and $G^T_{t,\tau}h_\tau$ is convergent in $L^1(P)$ when T goes to infinity for all $t \in [s,\tau]$. Then, for all $t \in [s,\tau]$,*

$$\rho^T_t(s, h_\tau) \xrightarrow{P} \rho^\infty_t(s, h_\tau), \qquad T \to +\infty.$$

As a result, ρ^∞ can be properly interpreted as a rate-adjusted price for the long run. The asset valuation through ρ^∞ exploits the stochastic discount factor long-term growth rate r^∞.

3.6 | Valuation of cashflows

Equation (14) describes the rate-adjusted price of a payoff h_τ in the time window $[s,T]$ with $s \leqslant \tau \leqslant T$. Disentangling the payment date τ from the horizon T permits to derive the rate-adjusted price for payoff streams. First, denote by $\alpha^T_t(s,h_\tau)$ the ratio between the rate-adjusted price and the no-arbitrage price of h_τ at any time $t \in [s,\tau]$:

$$\rho^T_t(s,h_\tau) = \alpha^T_t(s,h_\tau)\pi_t(h_\tau) = e^{-(r^T_s - r^\tau_t)(\tau-t)}\frac{\mathbb{E}^{F^T}_t[h_\tau]}{\mathbb{E}^{F^\tau}_t[h_\tau]}\pi_t(h_\tau). \tag{15}$$

Note that $\alpha^T_t(s,h_T)$ does not depend on h_T and it is consistent with Equation (11).

Consider now a cashflow $h = \{h_{\tau_i}\}_{i=1}^N$ with $N \in \mathbb{N}$, $s \leqslant \tau_1 < \cdots < \tau_N \leqslant T$ and $h_{\tau_i} \in L^1(\mathcal{F}_{\tau_i}, F^T)$ for all i. The rate-adjusted price $\rho^T_t(s,h)$ of the cashflow follows from standard linear pricing and the related rate adjustment is the ratio between $\rho^T_t(s,h)$ and the no-arbitrage price $\pi_t(h)$:

$$\rho^T_t(s,h) = \sum_{\tau_i \geqslant t} \rho^T_t\left(s, h_{\tau_i}\right) = \alpha^T_t(s,h)\pi_t(h), \tag{16}$$

$$\alpha^T_t(s,h) = \frac{\sum_{\tau_i \geqslant t}\alpha^T_t\left(s,h_{\tau_i}\right)\pi_t\left(h_{\tau_i}\right)}{\sum_{\tau_i \geqslant t}\pi_t\left(h_{\tau_i}\right)}. \tag{17}$$

The evolution of α^T, as well as the difference between ρ^T and π, can be useful for quantifying the interest rate risk exposure of the cashflow over time. Indeed, the no-arbitrage price of h decomposes into the product of ρ^T and the inverse of α^T. Since the rate-adjusted price captures the long-term interest rate risk of h (as we further illustrate in Subsection 4.2), the short-term interest rate exposure flows into the adjustment. This approach is fruitful for the risk management of securities that feature long maturities and sensitivity to shocks in the term structure of rates (see Section 5).

4 | PRICING KERNEL GROWTH

From problem (9), it is clear that $\mathcal{D}_s\rho^T$ is weakly time-differentiable in $[s, T]$. The same holds for weak time-derivatives of higher orders. As a result, the rate-adjusted price ρ^T is infinitely weakly time-differentiable and so it belongs to $\mathcal{U}_s^\infty$. A parallel reasoning ensures the infinite weak time-differentiability of ρ^∞.

Since $\mathcal{D}_s(\xi_s u) = \xi_s \mathcal{D}_s u$ for all $u \in \mathcal{U}_s^\infty$ and $\xi_s \in L^0(\mathcal{F}_s)$, the weak time-derivative in $[s, T]$ defines an L^0-linear operator $\mathcal{D}_s : \mathcal{U}_s^\infty \to \mathcal{U}_s^\infty$ and ρ^T satisfies the eigenvalue-eigenvector problem

$$\mathcal{D}_s\rho^T = r_s^T \rho^T, \tag{18}$$

where the eigenvalue belongs to $L^0(\mathcal{F}_s)$. Accordingly, ρ^∞ solves

$$\mathcal{D}_s\rho^\infty = r^\infty \rho^\infty \tag{19}$$

where the eigenvalue is a positive number.

As in Hansen and Scheinkman (2009), we suppose that the payoff h_T is positive so that ρ^T and ρ^∞ are positive. Hence, ρ^T and ρ^∞ are *principal eigenvectors* related to r_s^T and r^∞, respectively. This property is in line with the Perron–Frobenius theory, usually employed in Markov environments and successfully applied by Ross (2015) and Qin and Linetsky (2016). Indeed, when rates are constant, the principal eigenvalue associated with a differential price operator turns out to be the growth rate of the stochastic discount factor (Hansen & Scheinkman, 2009).

In this section, we analyze the relation between the eigenvalues of several differential pricing problems and pricing kernel growth rates in our stochastic-rate market. To build our theory, we introduce the notion of rate-adjusted pricing kernel. Proofs can be found in Appendix B.

4.1 | Finite- and infinite-horizon pricing kernel decomposition

The long-term eigenvalue problem of Equation (19) actually exploits the pricing kernel long-term growth rate, that is the long-term yield. Indeed, as shown by Qin and Linetsky (2017), under the Assumptions 2.1, $M_{s,t}$ satisfies the long-run decomposition of Equation (5), that is,

$$M_{s,t} = e^{-r^\infty(t-s)} \frac{b_s^\infty}{b_t^\infty} G_{s,t}^\infty.$$

The question, here, is whether an analogous property holds for the T-horizon problem of Equation (18). We need first to characterize the finite-horizon pricing kernel growth rate.

As introduced in Subsection 2.3, the pricing kernel $M_{s,t}$ satisfies

$$M_{s,t} = \frac{e^{-r_s^T(T-s)}}{e^{-r_t^T(T-t)}} G_{s,t}^T = \frac{e^{r_s^T s}\pi_s(1_T)}{\pi_s(1_{T-t})} \frac{e^{-r_s^T s}\pi_s(1_{T-t})}{\pi_t(1_T)} G_{s,t}^T$$

for all $T > t + s$. In the next proposition, we assess the asymptotic behaviors of the last three factors. Note that the ratio $\pi_s(1_T)/\pi_s(1_{T-t})$ is the return on a forward rate agreement between $T - t$ and T contracted at date s.

Proposition 4.1. *Under the Assumptions 2.1, for all $s > 0$ and $t > s$*

$$\frac{e^{r_s^T s}\pi_s(1_T)}{\pi_s(1_{T-t})} \xrightarrow{P} e^{-r^\infty(t-s)}, \qquad T \to +\infty$$
$$\frac{e^{-r_s^T s}\pi_s(1_{T-t})}{\pi_t(1_T)} \xrightarrow{P} \frac{b_s^\infty}{b_t^\infty}, \qquad T \to +\infty$$
$$G_{s,t}^T \xrightarrow{P} G_{s,t}^\infty, \qquad T \to +\infty.$$

The first convergence shows that, in addition to being a limit of bond yields, r^∞ is also a limit of continuously compounded forward rates contracted at s, as the convergence of $\pi_s(1_T)/\pi_s(1_{T-t})$ suggests. This approach is reminiscent of the constructions of Backus et al. (1989) and Alvarez and Jermann (2005). The second convergence involves the transient component of $M_{s,t}$. The limit of this factor is the ratio of the discounted values of the long bond. The third convergence regards the permanent (or martingale) component of the pricing kernel. At any finite horizon, this term consists of the Radon–Nikodym density of the T-forward measure. When the horizon is infinite, the long-term forward measure appears.

Consistently with the three convergences of Proposition 4.1, when the horizon is finite, we call growth term of $M_{s,t}$ the first ratio in the proposition, that is,

$$\frac{e^{r_s^T s}\pi_s(1_T)}{\pi_s(1_{T-t})} = e^{-r_s^T(T-2s)+r_s^{T-t}(T-t-s)}. \tag{20}$$

Hence, $M_{s,t}$ features a random growth term in $L^0(\mathcal{F}_s)$ determined by bond yields at time s. If rates are constant, this quantity actually reduces to $e^{-r(t-s)}$, which captures the growth of the pricing kernel $e^{-r(t-s)}L_{s,t}$.

As it is apparent from Equation (20), the growth rate of $M_{s,t}$ is not exactly r_s^T and so, differently from the long-term yield, it does not agree with the eigenvalue problem $\mathcal{D}_s\rho^T = r_s^T\rho^T$. Therefore, to isolate the growth rate r_s^T in finite horizons, we introduce the notion of *rate-adjusted pricing kernel*, which parallels the one of rate-adjusted prices.

4.2 | Rate-adjusted pricing kernel

Given any payoff h_T, we can write no-arbitrage and rate-adjusted prices at time t as

$$\pi_\tau(h_T) = \mathbb{E}_\tau\left[M_{\tau,T}h_T\right], \qquad \rho_\tau^T(s,h_T) = \mathbb{E}_\tau\left[N_{\tau,T}^T h_T\right],$$

where, for all τ, t in $[s,T]$ with $\tau \leqslant t$,

$$M_{\tau,t} = \frac{e^{r_t^T(T-t)}}{e^{r_\tau^T(T-\tau)}}G_{\tau,t}^T, \qquad N_{\tau,t}^T = \frac{e^{r_s^T(T-t)}}{e^{r_s^T(T-\tau)}}G_{\tau,t}^T.$$

We call $N_{s,t}^T$ the *rate-adjusted pricing kernel*. In particular, we have $N_{s,t}^T = e^{(r_s^T - r_t^T)(T-t)}M_{s,t}$ and it coincides with $M_{s,t}$ when rates are constant. The adjustment coefficient between $N_{s,t}^T$ and $M_{s,t}$ is the inverse of the one between π and ρ^T pointed out in Equation (11). In addition, the expected values of $M_{s,t}$ and $N^T_{s,t}$ differ only in the yields r^t_s and r^T_s:

$$\mathbb{E}_s\left[M_{s,t}\right] = e^{-r^t_s(t-s)}, \qquad \mathbb{E}_s\left[N^T_{s,t}\right] = e^{-r^T_s(t-s)}.$$

Since $N^T_{s,t} = e^{-r^T_s(t-s)}G^T_{s,t}$, its growth rate is r^T_s, in agreement with the rate-adjusted eigenvalue problem $\mathcal{D}_s\rho^T = r^T_s\rho^T$. Moreover, differently from $M_{s,t}$, the rate-adjusted pricing kernel is explicitly dependent on the horizon T under scrutiny. In the next proposition we establish the convergence of $N^T_{s,t}$ when T becomes arbitrarily large.

Proposition 4.2. *Under the Assumptions 2.1, for all $s > 0$ and $t > s$,*

$$N^T_{s,t} \xrightarrow{P} e^{-r^\infty(t-s)}G^\infty_{s,t}, \qquad T \to +\infty.$$

Therefore, we define the *long-term rate-adjusted pricing kernel* by

$$N^\infty_{s,t} = e^{-r^\infty(t-s)}G^\infty_{s,t} \tag{21}$$

and r^∞ naturally arises as growth rate for $N^\infty_{s,t}$.

A formalization of the fact that r^T_s and r^∞ are the rate-adjusted pricing kernel growth rates for finite (or infinite) horizons can be easily obtained in differential terms. When the horizon is finite, we can consider the problem

$$\begin{cases} \mathcal{D}_s f_t = -r^T_s\, f_t & t \in [s,T] \\ f_s = 1 \end{cases} \tag{22}$$

under the physical measure. When the horizon is infinite, we analyze (still under P)

$$\begin{cases} \mathcal{D}_s f_t = -r^\infty\, f_t & t \in [s,+\infty) \\ f_s = 1. \end{cases} \tag{23}$$

The last problem uses weak time-derivatives in $[s,+\infty)$ that can be defined by replacing T with $+\infty$ in Definition 2.2. The module $\mathcal{U}_s$ modifies accordingly by omitting the left-continuity of processes u at T but additionally requiring that $\int_s^{+\infty}\mathbb{E}_s[|u_\tau|]d\tau$ belongs to $L^0(\mathcal{F}_s)$.

Theorem 4.3. *Under the measure P, $\{N^T_{s,t}\}_t$ solves problem (22) in $\mathcal{U}^1_s$ with finite T and $\{N^\infty_{s,t}\}_t$ solves problem (23) in $\mathcal{U}^1_s$ with infinite T.*

The last theorem formalizes the growth terms r^T_s and r^∞ for the rate-adjusted pricing kernel at finite and infinite horizons. The results are consistent with the eigenvalue problems $\mathcal{D}\rho^T = r^T_s\rho^T$ and $\mathcal{D}\rho^\infty = r^\infty\rho^\infty$ satisfied by rate-adjusted prices.

Problems (22) and (23) generalize the differential relation $\mathcal{D}_s M = -rM$ which is satisfied by the pricing kernel when interest rates are constantly equal to r. When rates are stochastic, in standard diffusive models, the pricing kernel follows the dynamics $dM_{s,t} = -Y_t M_{s,t}dt - \nu_t M_{s,t}dW^P_t$, where ν_t is the market price of risk (see Subsection 6.2.2). The stochastic rate is present in the drift of the pricing kernel. However, any Y_t alone is not able to capture the growth rate of $M_{s,t}$ and it is not known ex ante. This last feature forbids any eigenvalue-eigenvector formulation of the problem of identifying a growth rate for $M_{s,t}$.

Problems (22) and (23) are not satisfied by the pricing kernel $M_{s,t}$ when interest rates are stochastic as well as no-arbitrage prices do not solve problems (9) and (12). The deep reason for these failures can be understood through the lens of the Hansen–Scheinkman decomposition.

By comparing Equation (21) with the long-term decomposition of the pricing kernel in Equation (5), it is apparent that $N^{\infty}_{s,t}$ features the same growth rate r^{∞} of $M_{s,t}$, as well as the same martingale component $G^{\infty}_{s,t}$. Nevertheless, the transitory component of $N^{\infty}_{s,t}$ is deterministic and equal to 1. Therefore, employing rate-adjusted prices for asset valuation means using a stochastic discount factor that is free from transitory effects in its long-term Hansen–Scheinkman decomposition. See also the previous Equation (6) that highlights the role of the transient component of $M_{s,t}$ in the price of any marketed payoff.

When rates are constant, the transient component of the stochastic discount factor is 1 and so the difference between $M_{s,t}$ and $N^{\infty}_{s,t}$ vanishes. On the contrary, when rates are stochastic, a trivial temporary term can be retrieved only in $N^{\infty}_{s,t}$. Rate-adjusted prices aggregate infinitesimal randomness to long-run risk exposure because $N^{T}_{s,t}$ is free from any transitory component. Bond yields are the proper financial variables that translate local riskiness to long-run risks through rate-adjusted prices. Table 1 summarizes our findings.

5 | APPLICATIONS OF RATE-ADJUSTED PRICING

The introduction of rate adjustments is fruitful in several areas of finance to disentangle the long- and short-term interest risk of cashflows.

5.1 | Interest rate swaps

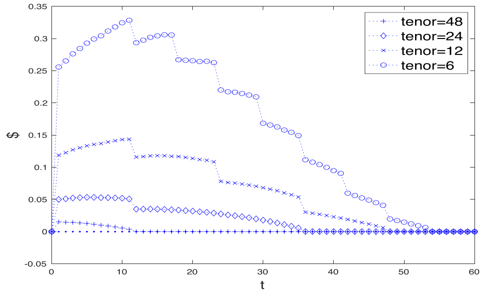
Consider an interest rate swap contract with periodic payment exchanges at dates $\tau_1, \dots, \tau_N$ in the time interval $[0, T]$ with $\tau_N = T$. The reference period for the first payment is $[\tau_0, \tau_1]$ with $0 < \tau_0 < \tau_1$. At any payment date τ_i, two parties exchange a fixed leg K with a floating leg made by the LIBOR rate $\mathcal{L}(\tau_{i-1}, \tau_i)$, which satisfies $(1 + (\tau_i - \tau_{i-1})\mathcal{L}(\tau_{i-1}, \tau_i))\pi_{\tau_{i-1}}(1_\tau) = 1$. Hence, the cashflow h of the swap payer is given by $h_{\tau_i} = (\mathcal{L}(\tau_{i-1}, \tau_i) - K)(\tau_i - \tau_{i-1})$ for any $i = 1, \dots, N$. Then, the no-arbitrage price of any h_{τ_i} at time $t \in [0, \tau_i]$ is

$$\pi_t\left(h_{\tau_i}\right) = \begin{cases} \pi_t\left(1_{\tau_{i-1}}\right) - \pi_t\left(1_{\tau_i}\right) - k\left(\tau_i - \tau_{i-1}\right)\pi_t\left(1_{\tau_i}\right) & t \in [0, \tau_{i-1}) \\ h_{\tau_i}\pi_t\left(1_{\tau_i}\right) & t \in [\tau_{i-1}, \tau_i] \end{cases}$$

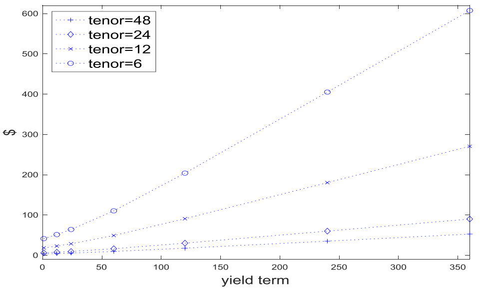
and the premium K is set in a way that $\pi_0(h) = 0$. We compute the no-arbitrage and the rate-adjusted price of the swap following Subsection 3.6.

We do a Monte Carlo simulation by assuming a Vasicek (1977) term structure of rates (see Subsection 6.1.1). We fix the first payment date τ_1 and the maturity T but we increase the number of exchanges in between. In so doing, we show that the rate-adjusted price relies mainly on the *maturity* of the swap, while the rate adjustment depends on the *tenor*. Indeed, short tenors are associated with a large short-term interest rate risk, which is much less relevant when payment exchanges are infrequent.

The main steps of the simulation are the following. From a random sample, we simulate several Wiener processes under the physical measure. We then simulate the Vasicek interest rate process, as well as ZCB prices, yields to maturity and the Radon–Nikodym derivatives L_T and



(a) Interest rate swap valuation.



(b) Interest rate swap DV01.

FIGURE 2 Left panel: difference between $\rho_t^T(0,h)$ and $\pi_t(h)$ of the interest rate swaps of Subsection 5.1 over time. Right panel: difference of the absolute value of the mean $DV01_t^\tau(\rho)$ and that of the mean $DV01_t^\tau(\pi)$ of the interest rate swaps, for increasing yield terms from 1 to 360 months. In both panels, we set $\tau_0 = 12$ and $\tau_N = 60$ months, we consider decreasing tenors of 48, 24, 12, and 6 months, and we assume a Vasicek term structure of rates on monthly basis. [Color figure can be viewed at wileyonlinelibrary.com]

G_T^T. After that, we determine K, the no-arbitrage prices $\pi_t(h_{\tau_i})$ of each swap payment at any t in $[0,T]$ and the no-arbitrage price $\pi_t(h)$ of the swap contract. By Equation (15), we compute $\alpha_t^T(0,h_\tau)$ for any swap payment and, from Equation (17), we obtain $\alpha_t^T(0,h)$ for the whole swap. We finally compute the rate-adjusted price $\rho_t^T(0,h)$ of the swap from Equation (16), take the difference between $\rho_t^T(0,h)$ and $\pi_t(h)$ at any t and average across scenarios. We repeat the algorithm for different tenors.

We plot in the left panel of Figure 2 the difference between $\rho^T(0,h)$ and $\pi(h)$ of the swap contracts for decreasing tenors of 48, 24, 12, and 6 months, after setting $\tau_0 = 12$ and $\tau_N = 60$ months. Discounting with a constant rate r_0^T makes a relatively small discrepancy from no-arbitrage pricing when a single payment at T is considered. However, the difference between the no-arbitrage and the rate-adjusted price of the swap increases when the tenor shortens (and the number of payments increases): short tenors are sensitive to temporary interest rate variations and this sensitivity is captured by the rate adjustment.

To capture the sensitivity of rate-adjusted prices with respect to changes in yields-to-maturity, we compute some conventional quantities used to evaluate interest rate risk. Following Coleman (2011), we consider partial $DV01$ (partial dollar durations or partial dollar values of an 01) with respect to yields with different maturities τ:

$$DV01_t^\tau(\pi) = -\frac{\partial \pi_t(h)}{\partial r_t^\tau}, \qquad DV01_t^\tau(\rho) = -\frac{\partial \rho_t^T(h)}{\partial r_t^\tau}.$$

These quantities correspond to the dollar change in price with respect to the variation of the considered yields. In our simulations, we focus on the increasing maturities of 1, 12, 24, 60, 120, 240, and 360 months. In the right panel of Figure 2, we plot the difference between the absolute value of the mean $DV01_t^\tau(\rho)$ and that of the mean $DV01_t^\tau(\pi)$, where the means are computed on the 60-month time span of the swap contract. A LOESS smoothing has been implemented before computing the increments. For any considered tenor, both $DV01_t^\tau(\pi)$ and $DV01_t^\tau(\rho)$ are increasing with respect to the yield term. In absolute values, average $DV01_t^\tau(\rho)$ are always higher than average $DV01_t^\tau(\pi)$. Moreover, the difference between the two increases as the tenor shortens. On the one hand, as we already observed, the interest rate risk exposures of the two swap prices are similar for the longest tenors. On the other, when the tenor shrinks, the dollar change in rate-adjusted swap prices with respect to the yields with long maturities is remarkably higher than that of the no-arbitrage swap prices. This behavior confirms the long-term nature of rate-adjusted prices.

5.2 | Whole life insurance actuarial present value

Consider a whole life insurance that pays a benefit equal to 1 when the subscriber dies. The contract has no preset horizon and the date of the unique payment is unknown ex ante. This date is related to the time-until-death, a random variable with probability density function $\wp$ on $[0, +\infty)$, that represents the difference between the insured's age at death and her age at the subscription. Subscribers with different ages feature different future lifetime, and so different $\wp$. See Chapters 3 and 4 in Bower et al. (1997).

The actuarial present value of the insurance is the expectation of the present value of the payment. This value summarizes the overall exposure of the insurance company to the obligation of the contract. Since the horizon of the insurance policy is potentially very large, the actuarial present value depends on the predicted term structure of rates. See, for example, Panjer and Bellhouse (1980) for the use of stochastic rates in the valuation of life contingencies. The introduction of rate adjustments additionally permits to disentangle the short- and long-term interest rate risk beared by the insurance company.

Suppose that the physical measure (used by the insurance company) coincides with the risk-neutral measure. In this case, in line with the previous notation, we can denote the actuarial present value by $\pi_0(h)$ and consider the *rate-adjusted* actuarial present value $\rho_0^\infty(0, h)$. Here h represents the unitary random payment of the whole life insurance (the subscriber's age is given):

$$\pi_0(h) = \int_0^{+\infty} e^{-r_0^\tau \tau} \wp(\tau) d\tau, \qquad \rho_0^\infty(0, h) = \int_0^{+\infty} e^{-r^\infty \tau} \wp(\tau) d\tau.$$

We assume that the time-until-death has cumulative distribution function $\mathcal{P}(\tau) = 1 - e^{-\gamma\tau^3}$ and in the simulations, we move the parameter γ from 0.00001 to 0.00059. The distribution is unimodal and increasing γ means moving the peak of time-until-death towards the date of the insurance subscription. This means considering shorter future lifetimes or, equivalently, increasing the subscriber's age. The used monthly time grid involves 900 months after the subscription for all γ. Interest rates are simulated following Vasicek (1977) with the specifications of Subsection 6.1.1.

Figure 3 represents the difference between π_0 and ρ_0^∞ when γ increases. The difference is small when γ is low and becomes sizable for high values of the parameter. The rate-adjusted actuarial present value differs more from the actuarial present value when old subscribers (high γ) are considered. Their future lifetime is short and so short-term rates have an important role in the computation of their actuarial present value. This valuation contrasts with the rate-adjusted one, which responds substantially to long-term interest rate shocks. The difference between the two actuarial values captures the short-term interest rate risk. On the contrary, ρ_0^∞ is more similar to π_0 for young insured (low γ) because their time-until-death is large and so their actuarial present value exploits heavily the long-run rates. Considering the rate, adjustment provides insights on the short and long-term interest rate risk exposure of the insurance company.

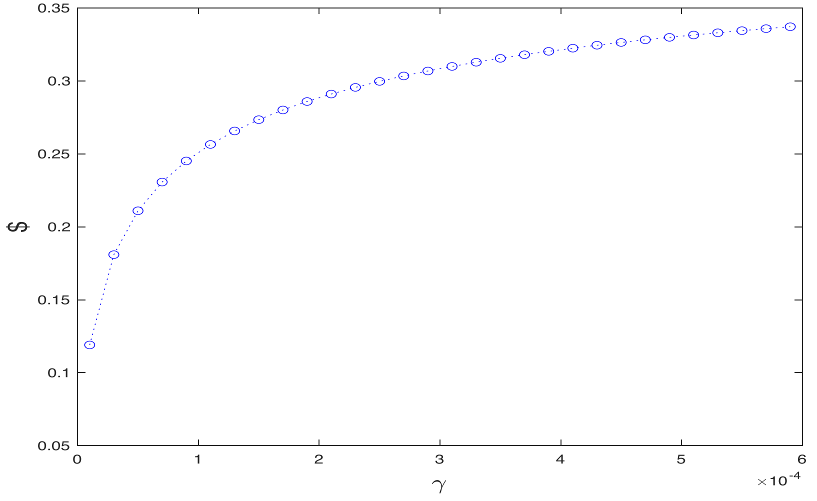


FIGURE 3 Difference between $\pi_0(h)$ and $\rho_0^\infty(0,h)$ of the whole life insurance of Subsection 5.2. The horizontal axis contains the parameter γ, which increases with the insured's age at subscription. A Vasicek term structure of rates is assumed on monthly basis. [Color figure can be viewed at wileyonlinelibrary.com]

6 | SINGLE-FACTOR AFFINE INTEREST RATE MODELS

We provide an illustration of our theory in arbitrage-free markets with diffusive short-term rates. In particular, we devote some attention to the Feynman–Kaç PDEs satisfied by rate-adjusted and no-arbitrage prices in Subsection 6.2.1. Subsection 6.1 considers a fixed-income market while Subsection 6.2 involves a market with both stocks and ZCBs.

6.1 | Pricing in a fixed-income market

We start comparing the drifts of ZCB prices under the measures P, Q, and F^T in a fixed-income market. Then, we make a comparison with rate-adjusted prices. When the processes under scrutiny are weakly time-differentiable, the drift is the weak time-derivative (Marinacci & Severino, 2018) and Theorem 3.2 applies. We use Itô's formula extensively, implicitly postulating that the processes under consideration are continuously differentiable with respect to time and twice continuously differentiable with respect to the spatial variable.

We assume that instantaneous rates follow the diffusion process $dY_t = \mu(t,Y_t)dt + \sigma(t,Y_t)dW_t^P$ in the time window $[s,T]$. Here, μ and σ are measurable functions of t and Y_t, and W_t^P denotes a Wiener process under the physical measure. The T-ZCB price at time t is function of t and Y_t and Itô's formula permits to determine its drift and diffusion coefficient:

$$\frac{d\pi_t(1_T)}{\pi_t(1_T)} = \tilde{\mu}(t,Y_t)dt + \tilde{\sigma}(t,Y_t)dW_t^P.$$

We indicate by ν the market price of risk process $\nu_t = (\tilde{\mu}(t,Y_t) - Y_t)/\tilde{\sigma}(t,Y_t)$. By Girsanov's theorem, we build a Wiener process under the risk-neutral measure Q via $dW_t^Q = dW_t^P + \nu_t dt$ and so

$$\frac{d\pi_t(1_T)}{\pi_t(1_T)} = Y_t\, dt + \tilde{\sigma}(t, Y_t) dW_t^Q.$$

Under Q, the drift coefficient of no-arbitrage bond prices is the instantaneous rate. When considering the forward measure, the dynamics of $\pi_t(1_T)$ become

$$\frac{d\pi_t(1_T)}{\pi_t(1_T)} = \left(Y_t + \tilde{\sigma}^2(t, Y_t)\right) dt + \tilde{\sigma}(t, Y_t) dW_t^{F^T}, \tag{24}$$

where $W_t^{F^T}$ is a Wiener process under F^T satisfying $dW_t^{F^T} = dW_t^Q - \tilde{\sigma}(t, Y_t)dt$. See Chapter 3 of Brigo and Mercurio (2006). Differently from the Q-dynamics, the drift under F^T depends on the diffusion coefficient, a feature that is absent in T-bond rate-adjusted prices because their differential satisfies

$$\frac{d\rho_t^T(s, 1_T)}{\rho_t^T(s, 1_T)} = r_s^T\, dt \tag{25}$$

for any considered measure. In agreement with Theorem 3.2, the weak time-derivative in $[s, T]$ of $\rho_t^T(s, 1_T)$ is $r_s^T \rho_t^T(s, 1_T)$, while under F^T, the candidate weak time-derivative of $\pi_t(1_T)$ is $(Y_t + \tilde{\sigma}^2(t, Y_t))\pi_t(1_T)$. Since $Y_t + \tilde{\sigma}^2(t, Y_t)$ is floating over time, under F^T, it is unlikely to find an eigenvalue-eigenvector formulation as problem (9) by using no-arbitrage prices instead of rate-adjusted prices when interest rates are stochastic.

6.1.1 | Zero-coupon bonds and call options in Vasicek (1977)

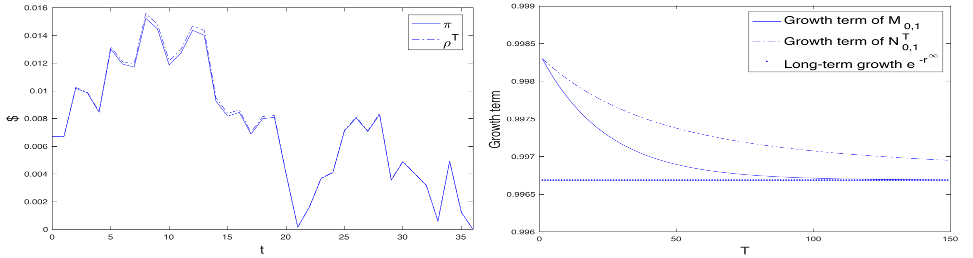
In single-factor affine term structure models—Vasicek (1977), Cox et al. (1985), and others—the price of a T-ZCB depends on the instantaneous rate Y_t through the exponential relation $\pi_t(1_T) = e^{A(t,T)-B(t,T)Y_t}$, with A and B deterministic functions. By Itô's differential rule, the diffusion coefficient of the no-arbitrage price $\pi_t(1_T)$ is $\tilde{\sigma}(t, Y_t) = -B(t, T)\sigma(t, Y_t)$. However, the dynamics of the related rate-adjusted price is still the one of Equation (25).

As an example, we consider the Vasicek (1977) model, in which the coefficients of the instantaneous rate are $\mu(t, Y_t) = k\theta - (k - \sigma\xi)Y_t$ and $\sigma(t, Y_t) = \sigma$, where $k, \theta, \sigma, \xi > 0$ and the market price of risk is ξY_t. Thus, the evolution of Y_t under Q is given by $dY_t = k(\theta - Y_t)\, dt + \sigma dW_t^Q$. The short-term rate is mean-reverting towards θ in the long run at a speed dictated by k. Under this specification,

$$A(t, T) = \left(\theta - \frac{\sigma^2}{2k^2}\right)(B(t, T) - T + t) - \frac{\sigma^2}{4k} B^2(t, T)$$

and $B(t, T) = (1 - e^{-k(T-t)})/k$. See Section 3.2 of Brigo and Mercurio (2006). The drift coefficient of the no-arbitrage bond price under F^T is $Y_t + \left(1 - e^{-k(T-t)}\right)^2 \sigma^2/k^2$. When the horizon T becomes infinitely large, this term converges a.s. to $Y_t + \sigma^2/k^2$. On the contrary, the drift parameter r_s^T of the rate-adjusted price converges a.s. to the long-term yield $r^\infty = \theta - \sigma^2/2k^2$.

To provide an example on derivative pricing, we consider a European call option with expiration τ and strike price c over a T-ZCB with $T > \tau$. Under Vasicek dynamics, the no-arbitrage price of



(a) Call option on a ZCB. (b) Growth terms.

FIGURE 4 Left panel: realizations of $\rho_t^\tau(0, h_\tau)$ and $\pi_t(h_\tau)$ for a European call option on a ZCB as described in Subsection 6.1.1. No-arbitrage prices are represented by solid lines and rate-adjusted prices by dashed lines. Right panel: term structure of growth terms of the pricing kernel $M_{0,1}$ and the rated-adjusted pricing kernel $N_{0,1}^T$ in Vasicek model. The solid line represents the growth term of $M_{0,1}$ for increasing horizons T, while the dashed line regards the growth term of $N_{0,1}^T$. The horizontal line is the long-term growth e^{-r^∞}. [Color figure can be viewed at wileyonlinelibrary.com]

the option at any time t between 0 and τ is $\pi_t(h_\tau) = \pi_t(1_T)\mathcal{N}(q) - c\pi_t(1_\tau)\mathcal{N}(q - \hat{\sigma})$, where $\mathcal{N}$ denotes the cumulative distribution function of a standard Gaussian,

$$q = \frac{\log(\pi_t(1_T)) - \log(c\pi_t(1_\tau)) + \hat{\sigma}^2/2}{\hat{\sigma}},$$

$$\hat{\sigma}^2 = \sigma^2 \frac{1 - e^{-2(k-\sigma\xi)(\tau-t)}}{2(k-\sigma\xi)} \frac{\left(1 - e^{-(k-\sigma\xi)(T-\tau)}\right)^2}{(k-\sigma\xi)^2}.$$

See Jamshidian (1989). The rate-adjusted price of the option can be easily obtained from the no-arbitrage price through Equation (11). We plot both price processes in the left panel of Figure 4, where we use the parameters $k = 0.5$, $\theta = 0.04$, $\sigma = 0.01$, $\xi = 0.2$, and $Y_0 = 0.02$ on a monthly time grid and we set $T = 72$ months, $\tau = 36$ months, and $c = 0.89$. The difference between the two quantities may be appreciated mainly at some intermediate months.

6.1.2 | Long-term relations for any payoff and pricing kernel growth

We now consider a generic attainable payoff $h_T \in L_s^1(\mathcal{F}_T, F^T)$ in a market with exponential affine interest rates. The ratio between the rate-adjusted price and the no-arbitrage price depends on the instantaneous rates at instants s and t:

$$\frac{\rho_t^T(s, h_T)}{\pi_t(h_T)} = e^{\{A(s,T) - B(s,T)Y_s\}\frac{T-t}{T-s} - A(t,T) + B(t,T)Y_t}.$$

Under Vasicek assumptions, the long-run relation between ρ^T and π of Proposition 3.4 can be determined explicitly and the limit depends on the speed parameter k. If k is high, the two prices are almost indistinguishable for large maturities:

$$\frac{\rho_t^T(s, h_T)}{\pi_t(h_T)} \xrightarrow{a.s.} e^{-\frac{1}{k}(Y_s - Y_t)}, \qquad T \to +\infty.$$

We finally focus on the pricing kernel between s and t in case bond yields are affine. The growth term of $M_{s,t}$ is an exponential function of Y_s, namely

$$\frac{e^{r_s^T s}\pi_s(1_T)}{\pi_s(1_{T-t})} = e^{A(s,T)\left(\frac{T-2s}{T-s}\right)-A(s,T-t)-\left\{B(s,T)\left(\frac{T-2s}{T-s}\right)-B(s,T-t)\right\}Y_s}.$$

When the terminal date T goes to infinity, this term converges a.s. to $e^{-r^{\infty}(t-s)}$, as expected. In the right panel of Figure 4, we display this convergence in a Vasicek model with $s = 0$, $t = 1$ and the parameters of Subsection 6.1.1. In addition, we graphically compare the growth term of $M_{0,1}$ with the one of the rate-adjusted pricing kernel $N_{0,1}^T$. The two terms share the same long-run limit.

6.2 | Pricing with stocks and bonds

The situation is more involved when the market is generated by a set of risky assets with prices $X_t^{(1)}, \dots X_t^{(N)}$, beyond ZCBs. The drifts of these no-arbitrage prices under F^T are additionally affected by the correlation between the instantaneous rate and the idiosyncratic random component of the asset under consideration (see, e.g., Rabinovitch, 1989). On the contrary, the drift coefficient of the rate-adjusted price of any security is always equal to the yield r_s^T. To further elucidate the issue, we borrow the dynamics of rates and stock prices from Appendix B of Brigo and Mercurio (2006).

We assume that short-term rates move as in Vasicek model in $[s, T]$, with the same dynamics as Subsection 6.1.1. Then, we consider a stock price X_t that follows a geometric Brownian motion with volatility $\eta > 0$, correlated with interest rates shocks. The instantaneous correlation parameter between the two underlying Wiener processes is ϕ. We can make the two sources of randomness orthogonal and consider, without loss of generality,

$$\begin{cases} dX_t = X_t Y_t \, dt + \eta X_t \left(\phi dW_t^Q + \sqrt{1-\phi^2} dZ_t^Q\right) \\ dY_t = k(\theta - Y_t)dt + \sigma dW_t^Q, \end{cases}$$

where W_t^Q and Z_t^Q are independent Wiener processes. Under F^T, we get

$$\begin{cases} dX_t = X_t\left(Y_t - \frac{\phi\sigma\eta}{k}\left(1-e^{-k(T-t)}\right)\right)dt + \eta X_t\left(\phi dW_t^{F^T} + \sqrt{1-\phi^2} dZ_t^{F^T}\right) \\ dY_t = \left(k(\theta - Y_t) - \frac{\sigma^2}{k}\left(1-e^{-k(T-t)}\right)\right)dt + \sigma dW_t^{F^T}, \end{cases}$$

where the correlation parameter ϕ impacts on the drift of X_t under F^T.

In the following, we derive the Feynman–Kaç PDEs satisfied by no-arbitrage and rate-adjusted prices in this market and the dynamics of the related pricing kernels.

6.2.1 | Feynman–Kaç partial differential equations

Consider a contingent claim $h_T \in L_S^1(\mathcal{F}_T, F^T)$, which is a continuous function of X_T and Y_T. We determine the Feynman–Kaç PDEs necessarily satisfied by the two prices. We assume that $\pi_t(h_T)$ and $\rho_t^T(s, h_T)$ are continuously differentiable with respect to t and twice continuously differentiable with respect to the variables $X_t = x$ and $Y_t = y$. See, for example, Karatzas and Shreve (1991), Section 4.4. The derivations are available upon request.

The discounted price $e^{-\int_s^t Y_\tau d\tau}\pi_t(h_T)$ is a Q-martingale. Therefore, by setting its drift equal to zero under Q, we get the Feynman–Kaç PDE for π:

$$\frac{\partial \pi}{\partial t} + xy\frac{\partial \pi}{\partial x} + k(\theta - y)\frac{\partial \pi}{\partial y} + \frac{\eta^2 x^2}{2}\frac{\partial^2 \pi}{\partial x^2} + \frac{\sigma^2}{2}\frac{\partial^2 \pi}{\partial y^2} + \phi\sigma\eta x\frac{\partial^2 \pi}{\partial x \partial y} = y\pi \tag{26}$$

with $\pi_T(h_T) = h_T$. In case, the correlation parameter ϕ is null and interest rates are constant, the usual Black–Scholes PDE arises.

Regarding the rate-adjusted price, since the forward price $e^{r_s^T(T-t)}\rho_t^T(s, h_T)$ is an F^T-martingale, we set its drift equal to zero under F^T. Then, the Feynman–Kaç PDE for ρ^T is

$$\frac{\partial \rho^T}{\partial t} + x\left(y - \frac{\phi\sigma\eta}{k}\left(1 - e^{-k(T-t)}\right)\right)\frac{\partial \rho^T}{\partial x} + \frac{\eta^2 x^2}{2}\frac{\partial^2 \rho^T}{\partial x^2} + \frac{\sigma^2}{2}\frac{\partial^2 \rho^T}{\partial y^2}$$
$$+ \left(k(\theta - y) - \frac{\sigma^2}{k}\left(1 - e^{-k(T-t)}\right)\right)\frac{\partial \rho^T}{\partial y} + \phi\sigma\eta x\frac{\partial^2 \rho^T}{\partial x \partial y} = r_s^T \rho^T \tag{27}$$

with $\rho_T^T = h_T$. The right-hand side in Equations (26) and (27) contains the instantaneous rate y for π and the yield r_s^T for ρ^T. The coefficients of spatial first-order derivatives are also different but the dissimilarity reduces when k increases. In addition, the coefficients of $\partial\pi/\partial x$ and $\partial\rho^T/\partial x$ coincide when ϕ is null. Hence, the disparity between π and ρ^T can also be seized through the solution of different parabolic PDEs.

6.2.2 | Pricing kernel dynamics

We now explicitly establish the evolution of $M_{s,t}$ and $N_{s,t}^T$ in our market. Beyond the money market account $\{e^{\int_s^t Y_\tau d\tau}\}_{t\in[s,T]}$, the prices of the assets that generate the market satisfy

$$\begin{cases} dX_t = X_t Y_t \, dt + \eta X_t\left(\phi dW_t^Q + \sqrt{1-\phi^2} dZ_t^Q\right) \\ d\pi_t(1_T) = \pi_t(1_T) Y_t dt - \pi_t(1_T) B(t,T)\sigma dW_t^Q, \end{cases}$$

where the dynamics of $\pi_t(1_T)$ are derived in Subsection 6.1.1. Under the physical measure,

$$\begin{cases} dX_t = X_t \mu_t^X dt + \eta X_t\left(\phi dW_t^P + \sqrt{1-\phi^2} dZ_t^P\right) \\ d\pi_t(1_T) = \pi_t(1_T) \mu_t^P dt - \pi_t(1_T) B(t,T)\sigma dW_t^P, \end{cases}$$

where μ_t^X and μ_t^P are adapted processes. They are related to the drifts under Q via the bivariate process of market price of risk $[\nu_t^W, \nu_t^Z]'$ such that $[dW_t^Q, dZ_t^Q]' = [\nu_t^W, \nu_t^Z]'dt + [dW_t^P, dZ_t^P]'$. By assuming that $\mu_t^P = (1 - \xi B(t,T)\sigma)Y_t$ for some $\xi > 0$, we obtain

$$\nu_t^W = \xi Y_t, \qquad \nu_t^Z = \frac{\mu_t^X - Y_t - \eta\phi\nu_t^W}{\eta\sqrt{1-\phi^2}},$$

where ν_t^W is in line with the usual approach to Vasicek short-term rates.

Now consider the pricing kernel $M_{s,t}$. Since the processes defined by $M_{s,t}H_t$, where H_t is each of X_t, $\pi_t(1_T)$ and $e^{\int_s^t Y_\tau d\tau}$, are conditional P-martingales in $[s,T]$, their drifts are null and the dynamics of $M_{s,t}$ turns out to be

$$dM_{s,t} = -Y_t M_{s,t} dt - \nu_t^W M_{s,t} dW_t^P - \nu_t^Z M_{s,t} dZ_t^P.$$

The differential of the rate-adjusted pricing kernel $N_{s,t}^T$ can be inferred from the multiplicative relation $N_{s,t}^T = \pi_t(1_T) e^{r_s^T (T-t)} M_{s,t}$ by Itô's product rule:

$$dN_{s,t}^T = -r_s^T N_{s,t}^T dt - \left(\nu_t^W + B(t,T)\sigma\right) N_{s,t}^T dW_t^P - \nu_t^Z N_{s,t}^T dZ_t^P.$$

As expected, the dynamics of $N_{s,t}^T$ coincide with the ones of $M_{s,t}$ when interest rates are constant. Indeed, σ is null and the yield r_s^T coincides with the short-term rate. In general, the drift of N_s^T is driven by r_s^T in agreement with Theorem 4.3, while the one of $M_{s,t}$ exploits the stochastic rate Y_t.

7 | CONCLUSIONS

This paper generalizes to stochastic-rate markets a key property of risk-neutral pricing. Indeed, if interest rates are constant (and deterministic) over time, the instantaneous rate is both the principal eigenvalue in the return-rate relation and the stochastic discount factor growth rate. If rates are stochastic, this feature is satisfied by rate-adjusted prices that, in fact, are indistinguishable from no-arbitrage prices when rates are constant. In particular, ZCB yields replace instantaneous rates and the forward measure is employed instead of the risk-neutral one. Importantly, bond yields are able to capture the growth rate of rate-adjusted pricing kernels. This rate coincides with the one of the effective pricing kernel when the horizon under consideration is infinite. Moreover, rate-adjusted pricing kernels feature a trivial transient component in their Hansen–Scheinkman decomposition. These two facts allow us to link rate-adjusted prices to long-term interest rate risk.

The multiplicative decomposition of the no-arbitrage price of a security into a rate-adjusted price and a rate adjustment permits to disentangle the long- from the short-term exposure to interest rate risk. The rate-adjusted price is associated with persistent shocks in the term structure of rates, while the adjustment captures temporary variations in the yield curve. Therefore, our theory is fruitful for the risk management of financial contracts that feature long maturities as well as interest rate risk exposure. Beyond the applications in Section 5, our framework may shed some light on the maturity mismatch between deposits and loans in the banking system (Hoffmann et al., 2019), as well as the discounting methodology of life insurance and pension companies' assets and liabilities that are subject to the European Solvency II regulation (Jørgensen, 2018). In fact, such regulation assumes specific dynamics of rates based on the long-term yield (the Ultimate Forward Rate) dictated by European authorities every year. See the Smith and Wilson (2001) procedure.

An interesting avenue of future research is the relaxation of some of the Assumptions 2.1, to allow for stochastic long-term yields. Indeed, some discrepancies between the European regulatory defined yield curves and empirical data have been highlighted by Balter et al. (2021). Given also the fact that the Ultimate Forward Rate is regularly updated by the European authorities, Gouriéroux et al. (2023) introduce stochastic long-term rates by considering a sequence of arbitrage free markets with increasing maximal terms. Importantly, they distinguish between long-run interest rates, *ultra long run* term structure and an *ultimate* interest rate.

Considering further specific dynamics of interest rates may constitute an additional avenue for future research. An interesting approach to investigate is the one Brennan and Schwartz (1979), where the instantaneous rate and a long-term interest rate are modeled separately via a joint Gaussian diffusive process. Models with more interest rate factors, extracted for instance, from Principal Component Analysis, may be considered, too: see the seminal work of Litterman and Scheinkman (1991) and the literature review in Severino et al. (2022). In addition, it could also be desirable to characterize the evolution of short rates through exogenous factors that determine the information structure, in order to assess the relation between these factors and the interest rate risk exposure over time.

Finally, from a theoretical perspective, we studied eigenvalue-eigenvector problems in which the eigenvalues are random variables, known at the beginning of the time interval. A last challenge is to study the implications of random eigenvalues in the Perron–Frobenius theory that underlies the pricing kernel decomposition in Hansen and Scheinkman (2009). Indeed, random dominating eigenvalues may be an indicator of nondeterministic steady states for economic dynamics.

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DATA AVAILABILITY STATEMENT

Data employed to create Figure 1 are available on the https://www.federalreserve.gov/data/yield-curve-tables/feds200628_1.html FED website.

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APPENDIX A: ADDITIONAL THEORETICAL ISSUES

A.1 | Technical assumptions

In the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, the filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ satisfies the usual conditions and is left-continuous at T. We mean that $\mathbb{F}$ is complete and right-continuous, that is. $\mathcal{F}_t = \mathcal{F}_{t^+}$ for all $t \in [0, T)$, and $\mathcal{F}_T = \mathcal{F}_{T^-}$.

In the whole paper, we identify random variables that coincide almost surely and we identify stochastic processes up to modifications. Moreover, we consider processes $u : [0, T] \times \Omega \to \mathbb{R}$ that are adapted on the given filtered probability space. This requirement is equivalent to progressively measurability up to modifications (Proposition 1.12 in Karatzas & Shreve, 1991).

A.2 | Forward measures

The use of different numéraires is a common practice in asset pricing: see, for example. the comprehensive treatment by Geman et al. (1995). Regarding the T-forward measure, by Theorem 1 and Example 2 in Geman et al. (1995), the Radon–Nikodym derivative of F^T with respect to the risk-neutral measure Q on $\mathcal{F}_T$ is

$$J_T^T = \frac{dF^T}{dQ} = e^{-\int_0^T Y_\tau d\tau}\left(\mathbb{E}\left[L_T e^{-\int_0^T Y_\tau d\tau}\right]\right)^{-1} = e^{r_0^T T - \int_0^T Y_\tau d\tau}.$$

We also define, for any $t \in [0, T]$,

$$J_t^T = \mathbb{E}_t\left[L_{t,T} J_T^T\right] = e^{r_0^T T - r_t^T (T-t) - \int_0^t Y_\tau d\tau}$$

and we set $J_{t,T}^T = J_T^T / J_t^T$. The Radon–Nikodym derivative of F^T with respect to P on $\mathcal{F}_T$ is, then, $G_T^T = dF^T/dP = J_T^T L_T$ and we define, for any $t \in [0, T]$,

$$G_t^T = \mathbb{E}_t\left[G_T^T\right] = \mathbb{E}_t\left[L_T J_T^T\right] = L_t J_t^T.$$

As for t-bond yields, their relation with T-bond yields is expressed by the following compounding rule, which also ensures that $\mathbb{E}_s[M_{s,t}] = e^{-r_s^t(t-s)}$.

Lemma A.1. *For any $s \leqslant t \leqslant T$, we have* $e^{r_s^T(T-s)} = e^{r_s^t(t-s)}\mathbb{E}_s^{F^T}[e^{r_t^T(T-t)}]$.

A.3 | Weak time-derivative in $[s, T]$

Consider the conditional space $L_s^1(\mathcal{F}_T) = \{f \in L^0(\mathcal{F}_T) : \mathbb{E}_s[|f|] \in L^0(\mathcal{F}_s)\}$. Cerreia-Vioglio et al. (2016) show that $L_s^1(\mathcal{F}_T)$ is an L^0-module with the multiplicative decomposition $L_s^1(\mathcal{F}_T) = L^0(\mathcal{F}_s)L^1(\mathcal{F}_T)$. Clearly, $L_s^1(\mathcal{F}_T)$ contains all functions f in $L^1(\mathcal{F}_T)$: in this case $\mathbb{E}_s[|f|] \in L^1(\mathcal{F}_s)$. In general, however, the conditional expectation is defined for random variables that are merely in $L^0(\mathcal{F}_T)$. See Chapter II, §7 of Shiryaev (1996).

In $L_s^1(\mathcal{F}_T)$, we use the L^0-valued metric $d(f, g) = \mathbb{E}_s[|f - g|]$. Accordingly, we say that a stochastic process $u : [s, T] \to L_s^1(\mathcal{F}_T)$ is L_s^1-continuous if and only if, for all $t \in [s, T]$, $\mathbb{E}_s[|u_\tau - u_t|] \to 0$ a.s. when $\tau \to t$. This property is weaker than standard L^1-continuity.

Now consider the L^0-module $\mathcal{U}_s$. As a consequence of Tonelli's theorem, all processes in $\mathcal{U}_s$ are such that $\int_s^T \mathbb{E}_s[|u_\tau|]d\tau$ belongs to $L^0(\mathcal{F}_s)$, where the integral is computed trajectory by trajectory. In addition, $\mathcal{U}_s$ includes all conditional (or generalized) martingales.

We now focus on weak time-differentiability in $[s, T]$. The space $C_c^1((t, T), L^0(\mathcal{F}_s))$ employed in Definition 2.2 consists of functions $\varphi_s : [t, T] \to L^0(\mathcal{F}_s)$ that have compact support in (t, T) and are continuously differentiable over time in the following sense: there exists a continuous function $\psi : [t, T] \to L^0(\mathcal{F}_s)$ with compact support in (t, T) such that the pathwise integral $\int_t^\tau \psi(z)dz$ equals $\varphi_s(\tau)$ for all $\tau \in [t, T]$. For simplicity, we denote ψ by φ_s'.

The weak time-derivative in $[s, T]$ is unique, up to modifications.

Proposition A.2. *Let $u \in \mathcal{U}_s$ be weakly time-differentiable in $[s, T]$. Then, the weak time-derivative of u in $[s, T]$ is unique.*

Proof. Follow the proof of Proposition 2.2 in Marinacci and Severino (2018) by replacing the unconditional expectation with the conditional expectation with respect to $\mathcal{F}_s$, and the convergence in L^1 with that in L_s^1. □

The next results give the relation between the weak time-derivatives in $[s_1, T]$ and $[s_2, T]$ for $s_1 \leqslant s_2$, and the characterization of conditional martingales.

Proposition A.3. *Let $0 \leqslant s_1 \leqslant s_2 \leqslant T$. If $u \in \mathcal{U}_{s_1}^1 \cap \mathcal{U}_{s_2}^1$, then $(\mathcal{D}_{s_1}u)_t = (\mathcal{D}_{s_2}u)_t$ for every $t \in [s_2, T]$.*

Proof. Since $u \in \mathcal{U}_{s_1}^1 \cap \mathcal{U}_{s_2}^1$, by Definition 2.2, for every $t \in [s_2, T]$,

$$\int_t^T \mathbb{E}_{s_i}\left[\left(\mathcal{D}_{s_i}u\right)_\tau \mathbf{1}_{A_t}\right]\varphi_{s_i}(\tau)d\tau = -\int_t^T \mathbb{E}_{s_i}\left[u_\tau \mathbf{1}_{A_t}\right]\varphi_{s_i}'(\tau)d\tau$$

for all $A_t \in \mathcal{F}_t, \varphi_{s_i} \in C_c^1((t, T), L^0(\mathcal{F}_{s_i}))$ and $i = 1, 2$. Since $\mathcal{F}_{s_1} \subset \mathcal{F}_{s_2}$, for any $t \in [s_2, T]$, $C_c^1((t, T), L^0(\mathcal{F}_{s_1})) \subset C_c^1((t, T), L^0(\mathcal{F}_{s_2}))$. In the following chain of equalities, we first exploit the weak time-differentiability of u in $[s_1, T]$ and then the one in $[s_2, T]$. For every $t \in [s_2, T]$, $A_t \in \mathcal{F}_t$, and $\varphi_{s_1} \in C_c^1((t,T), L^0(\mathcal{F}_{s_1}))$, we have

$$\int_t^T \mathbb{E}_{s_1}\left[\left(\mathcal{D}_{s_1}u\right)_\tau \mathbf{1}_{A_t}\right]\varphi_{s_1}(\tau)d\tau = -\int_t^T \mathbb{E}_{s_1}\left[u_\tau \mathbf{1}_{A_t}\right]\varphi'_{s_1}(\tau)d\tau$$

$$= -\int_t^T \mathbb{E}_{s_1}\left[\mathbb{E}_{s_2}\left[u_\tau \mathbf{1}_{A_t}\right]\right]\varphi'_{s_1}(\tau)d\tau = \mathbb{E}_{s_1}\left[-\int_t^T \mathbb{E}_{s_2}\left[u_\tau \mathbf{1}_{A_t}\right]\varphi'_{s_1}(\tau)d\tau\right]$$

$$= \mathbb{E}_{s_1}\left[\int_t^T \mathbb{E}_{s_2}\left[\left(\mathcal{D}_{s_2}u\right)_\tau \mathbf{1}_{A_t}\right]\varphi_{s_1}(\tau)d\tau\right] = \int_t^T \mathbb{E}_{s_1}\left[\left(\mathcal{D}_{s_2}u\right)_\tau \mathbf{1}_{A_t}\right]\varphi_{s_1}(\tau)d\tau.$$

The uniqueness of the weak time-derivative in $[s_1, T]$ implies that $(\mathcal{D}_{s_1}u)_t = (\mathcal{D}_{s_2}u)_t$ for every $t \in [s_2, T]$. □

Proposition A.4. *A process u belongs to $\mathcal{U}_s^1$ and has $\mathcal{D}_s u = 0$ if and only if it is a conditional martingale.*

Proof. Suppose that u is a conditional martingale. Then, u belongs to $\mathcal{U}_s$. Moreover, fixed $t \in [s,T]$, for any $\varphi_s \in C_c^1((t,T), L^0(\mathcal{F}_s))$ and $A_t \in \mathcal{F}_t$,

$$\int_t^T \mathbb{E}_s\left[u_\tau \mathbf{1}_{A_t}\right]\varphi'_s(\tau)d\tau = \int_t^T \mathbb{E}_s\left[u_t \mathbf{1}_{A_t}\right]\varphi'_s(\tau)d\tau = \mathbb{E}_s\left[u_t \mathbf{1}_{A_t}\right]\int_t^T \varphi'_s(\tau)d\tau = 0$$

because φ_s is in $C_c^1((t,T), L^0(\mathcal{F}_s))$. Hence, $w(t) = 0$ for any $t \in [s,T]$ satisfies the definition of weak time-derivative of u in $[s,T]$ and so $\mathcal{D}_s u = 0$.

Conversely, assume that $u \in \mathcal{U}_s^1$ has $\mathcal{D}_s u = 0$. First, u is adapted and any $u_\tau \in L_s^1(\mathcal{F}_\tau)$. Therefore, $\mathbb{E}_s[|u_\tau|] \in L^0(\mathcal{F}_s)$ for all $\tau \in [s,T]$. As a consequence, $\mathbb{E}_t[|u_\tau|] \in L^0(\mathcal{F}_t)$ for all $s \leqslant t \leqslant \tau \leqslant T$. Indeed, since $|u_\tau|$ is non-negative, $\mathbb{E}_t[|u_\tau|]$ is always defined as $\mathcal{F}_t$-measurable extended random variable. However, if there existed a set $A_t \in \mathcal{F}_t$ such that $\mathbb{E}_t[|u_\tau|]\mathbf{1}_{A_t}$ equals infinity, then, taken any $B_s \in \mathcal{F}_s$ with nonempty $A_t \cap B_s$, $\mathbb{E}_s[|u_\tau|\mathbf{1}_{B_s}] \geqslant \mathbb{E}_s[\mathbb{E}_t[|u_\tau|]\mathbf{1}_{A_t \cap B_s}]$, which is infinite. This fact would contradict $u_\tau \in L_s^1(\mathcal{F}_\tau)$.

Hence, in order to prove that u is a conditional martingale, we are just left to show that u satisfies the martingale property. We begin with proving that, given $t \in [s,T]$, $\mathbb{E}_t[u_\tau]$ is not dependent on τ for a.e. $\tau \in [t,T]$.

Take into consideration a continuous function $\eta : [t,T] \to \mathbb{R}$ with compact support in (t,T) such that $\int_t^T \eta(\tau)d\tau = 1$. Given a continuous function $\xi : [t,T] \to \mathbb{R}$ with compact support in (t,T), we define the function $k_\xi : [t,T] \to \mathbb{R}$ by $k_\xi(\sigma) = \xi(\sigma) - (\int_t^T \xi(\tau)d\tau)\eta(\sigma)$.

The function k_ξ is continuous with compact support in (t,T) and $\int_t^T k_\xi(\tau)d\tau = 0$. Thus, k_ξ has a primitive K_ξ that is continuous with compact support in (t,T). Since $K_\xi \in C_c^1((t,T), \mathbb{R})$, it is included in $C_c^1((t,T), L^0(\mathcal{F}_s))$ and so we use it as a test function in the definition of weak time-derivative of u in $[s,T]$. Since $\mathcal{D}_s u = 0$, for any $A_t \in \mathcal{F}_t$, the following holds:

$$0 = \int_t^T \mathbb{E}_s\left[u_\sigma \mathbf{1}_{A_t}\right]\left(\xi(\sigma) - \left(\int_t^T \xi(\tau)d\tau\right)\eta(\sigma)\right)d\sigma$$

$$= \int_t^T \mathbb{E}_s\left[u_\sigma \mathbf{1}_{A_t}\right]\xi(\sigma)d\sigma - \int_t^T \mathbb{E}_s\left[u_\sigma \mathbf{1}_{A_t}\right]\left(\int_t^T \xi(\tau)d\tau\right)\eta(\sigma)d\sigma$$

$$= \int_t^T \mathbb{E}_s\left[u_\tau \mathbf{1}_{A_t}\right]\xi(\tau)d\tau - \int_t^T \left(\int_t^T \mathbb{E}_s\left[u_\sigma \mathbf{1}_{A_t}\right]\eta(\sigma)d\sigma\right)\xi(\tau)d\tau$$

$$= \int_t^T \left(\mathbb{E}_s\left[u_\tau \mathbf{1}_{A_t}\right] - \int_t^T \mathbb{E}_s\left[u_\sigma \mathbf{1}_{A_t}\right]\eta(\sigma)d\sigma\right)\xi(\tau)d\tau.$$

By Lemma A.1 in the Appendix of Marinacci and Severino (2018), for a.e. $\tau \in [t,T]$, we have $\mathbb{E}_s\left[u_\tau \mathbf{1}_{A_t}\right] = \int_t^T \mathbb{E}_s\left[u_\sigma \mathbf{1}_{A_t}\right]\eta(\sigma)d\sigma$. Since $\int_t^T \eta(\sigma)d\sigma = 1$,

$$\int_t^T \left(\mathbb{E}_s\left[u_\tau \mathbf{1}_{A_t}\right] - \mathbb{E}_s\left[u_\sigma \mathbf{1}_{A_t}\right]\right)\eta(\sigma)d\sigma = 0.$$

As the last equality is satisfied by any continuous function η with compact support in (t,T), it follows that, for a.e. $\sigma \in [t,T]$, $\mathbb{E}_s[u_\sigma \mathbf{1}_{A_t}] = \mathbb{E}_s[u_\tau \mathbf{1}_{A_t}]$ and so $\mathbb{E}_t[u_\sigma] = \mathbb{E}_t[u_\tau]$. Consequently, $\mathbb{E}_t[u_\tau]$ is not dependent on τ for a.e. $\tau \in [t,T]$ and so $\mathbb{E}_t[u_\tau] = f_t$ for some $f_t \in L_s^1(\mathcal{F}_t)$.

u is L_s^1-right-continuous and so $\mathbb{E}_t[u_\tau]$ goes to u_t in L_s^1 when $\tau \to t^+$. Since for a.e. $\tau \in [t,T]$, $\mathbb{E}_t[u_\tau]$ coincides a.s. with f_t, which does not depend on τ, the uniqueness of the L_s^1-limit ensures that $f_t = u_t$. Therefore, for any $t \in [0,T]$ and for a.e. $\tau \in [t,T]$, $\mathbb{E}_t[u_\tau] = u_t$.

The last property is actually satisfied by any $\tau \in [t,T]$. Indeed, fix any τ and consider a sequence $\{\tau_i\}_{i\in\mathbb{N}} \subset [t,T]$ such that $\tau_i \to \tau^+$ and $\mathbb{E}_t[u_{\tau_i}] = u_t$. Since u is L_s^1-right-continuous, the L_s^1-limit of $\mathbb{E}_t[u_{\tau_i}]$ is $\mathbb{E}_t[u_\tau]$. Nevertheless, $\mathbb{E}_t[u_{\tau_i}] = u_t$ for all i and so, by uniqueness of the L_s^1-limit, $\mathbb{E}_t[u_\tau] = u_t$. □

APPENDIX B: PROOFS

B.1 | Proof of Theorem 3.2

(Existence) In order to show that $\rho^T \in \mathcal{U}_s^1$, we prove that $e^{r_s^T T}\rho^T$ belongs to $\mathcal{U}_s$ and is weakly time-differentiable in $[s,T]$.

First, for all $\tau \in [s,T]$, $e^{r_s^T T}\rho_\tau^T \in L_s^1(\mathcal{F}_\tau)$. Indeed, $|e^{r_s^T T}\rho_\tau^T|$ is non-negative and so its conditional expectation at time s is an extended real random variable. However, $\mathbb{E}_s^{F^T}[|e^{r_s^T T}\rho_\tau^T|] \leqslant e^{r_s^T \tau}\mathbb{E}_s^{F^T}[|h_T|]$, which is in $L^0(\mathcal{F}_s)$. Thus, $\mathbb{E}_s^{F^T}[|e^{r_s^T T}\rho_\tau^T|]$ is in $L^0(\mathcal{F}_s)$.

Regarding L_s^1-continuity, we check that, for any $t \in [s,T)$, $\mathbb{E}_s^F[|e^{r_s^T T}\rho_\tau^T - e^{r_s^T T}\rho_t^T|]$ tends to zero when $\tau \to t^+$. We have

$$\mathbb{E}_s^{F^T}\left[\left|e^{r_s^T T}\rho_\tau^T - e^{r_s^T T}\rho_t^T\right|\right] = e^{r_s^T t}\mathbb{E}_s^{F^T}\left[\left|e^{r_s^T(\tau-t)}\mathbb{E}_\tau^{F^T}[h_T] - \mathbb{E}_t^{F^T}[h_T]\right|\right]$$

$$\leqslant e^{r_s^T t}\left(\mathbb{E}_s^{F^T}\left[\left|e^{r_s^T(\tau-t)}\mathbb{E}_\tau^{F^T}[h_T] - \mathbb{E}_\tau^{F^T}[h_T]\right|\right]\right.$$

$$\left. + \mathbb{E}_s^{F^T}\left[\left|\mathbb{E}_\tau^{F^T}[h_T] - \mathbb{E}_t^{F^T}[h_T]\right|\right]\right)$$

$$\leqslant e^{r_s^T t}\left(\left|e^{r_s^T(\tau-t)} - 1\right|\mathbb{E}_s^{F^T}[|h_T|] + \mathbb{E}_s^{F^T}\left[\left|\mathbb{E}_\tau^{F^T}[h_T] - \mathbb{E}_t^{F^T}[h_T]\right|\right]\right).$$

In the last expression, both addends go to zero a.s. when τ approaches t^+. In particular, the convergence of the first one follows from the fact that almost every realization of r_s^T is a real number (fixed for the convergence). As to the second term, its convergence is ensured by Lévy's downward theorem that guarantees that $\mathbb{E}_\tau^{F^T}[h_T]$ goes in L_s^1 to $\mathbb{E}_{t^+}^{F^T}[h_T] = \mathbb{E}_t^{F^T}[h_T]$ when $\tau \to t^+$. Similarly, when $\tau \to T^-$, the convergence is due to Lévy's upward theorem. Therefore, $e^{r_s^T T}\rho^T$ belongs to $\mathcal{U}_s$. To be precise, as described in Chapter 14 of Williams (1991), Lévy's theorems require that $h_T \in L^1(\mathcal{F}_T)$ and ensure the previous convergences in L^1 of conditional expectations. However, these results still hold when $h_T \in L_s^1(\mathcal{F}_T)$ by using the convergence in L_s^1. Indeed, as shown in Cerreia-Vioglio et al. (2016), $L_s^1(\mathcal{F}_T) = L^0(\mathcal{F}_s)L^1(\mathcal{F}_T)$ and so $h_T = a_s k_T$ for some $a_s \in L^0(\mathcal{F}_s)$ and $k_T \in L^1(\mathcal{F}_T)$. Then, Lévy's theorems apply to k_T with the convergence in L^1. The latter implies the convergence in L_s^1, which holds also after multiplying k_T by a_s and retrieving h_T.

Now we look for the weak time-derivative in $[s,T]$ of $e^{r_s^T T}\rho^T$. We consider any $A_t \in \mathcal{F}_t$ and $\varphi_s \in C_c^1((t,T), L^0(\mathcal{F}_s))$. Since indicator functions $\mathbf{1}_{A_t}$ are $\mathcal{F}_\tau$-measurable for all $\tau \in [t,T]$,

$$
\begin{aligned}
&-\int_t^T \mathbb{E}_s^{F^T}\left[e^{r_s^T T}\rho_\tau^T \mathbf{1}_{A_t}\right]\varphi_s'(\tau)d\tau = -\int_t^T e^{r_s^T \tau}\mathbb{E}_s^{F^T}\left[h_T \mathbf{1}_{A_t}\right]\varphi'(\tau)d\tau \\
&= -\mathbb{E}_s^{F^T}\left[h_T \mathbf{1}_{A_t}\right]\int_t^T e^{r_s^T \tau}\varphi_s'(\tau)d\tau = \mathbb{E}_s^{F^T}\left[h_T \mathbf{1}_{A_t}\right]\int_t^T r_s^T e^{r_s^T \tau}\varphi_s(\tau)d\tau \\
&= \int_t^T r_s^T \mathbb{E}_s^{F^T}\left[e^{r_s^T \tau} h_T \mathbf{1}_{A_t}\right]\varphi_s(\tau)d\tau = \int_t^T \mathbb{E}_s^{F^T}\left[r_s^T e^{r_s^T T}\rho_\tau^T \mathbf{1}_{A_t}\right]\varphi_s(\tau)d\tau.
\end{aligned}
$$

The integral of the function $\sigma \mapsto e^{r_s^T \sigma}\varphi_s'(\sigma)$ is computed pathwise in $L^0(\mathcal{F}_s)$, exploiting the compact support of φ_s.

Therefore, the candidate weak time-derivative in $[s,T]$ of $e^{r_s^T T}\rho^T$ is $r_s^T e^{r_s^T T}\rho^T$. Since $r_s^T e^{r_s^T T}\rho^T$ belongs to $\mathcal{U}_s$, we can claim that $\mathcal{D}_s\rho^T = r_s^T\rho^T$. Of course, $\rho_T^T = h_T$ and so $\rho^T \in \mathcal{U}_s^1$ solves problem (9).

(Uniqueness) Let $f^{(1)}, f^{(2)} \in \mathcal{U}_s^1$ be two solutions of problem (9) and define $z = f^{(1)} - f^{(2)} \in \mathcal{U}_s^1$. We have that $\mathcal{D}_s z = r_s^T z$ and $z_T = 0$. We now compute the weak time-derivative of $e^{-rt}z_t$ in $[s,T]$. Fix $t \in [s,T]$. For any $\varphi_s \in C_c^1((t,T), L^0(\mathcal{F}_s))$, consider the function $\theta \mapsto e^{-r_s^T\theta}r_s^T\varphi_s(\theta) - e^{-r_s^T\theta}\varphi_s'(\theta)$ that takes values in $L^0(\mathcal{F}_s)$. By integrating pathwise, it follows that

$$
\int_\tau^T \left(e^{-r_s^T\theta}r_s^T\varphi_s(\theta) - e^{-r_s^T\theta}\varphi_s'(\theta)\right)d\theta = e^{-r_s^T\tau}\varphi_s(\tau).
$$

Hence, $e^{-r_s^T\tau}\varphi_s(\tau)$ belongs to $C_c^1((t,T), L^0(\mathcal{F}_s))$ and so we can use it as test function in the definition of weak time-derivative of z in $[s,T]$:

$$
\begin{aligned}
&\int_t^T \mathbb{E}_s^{F^T}\left[\mathcal{D}_s z_\tau \mathbf{1}_{A_t} e^{-r_s^T\tau}\right]\varphi_s(\tau)d\tau \\
&= -\int_t^T \mathbb{E}_s^{F^T}\left[z_\tau \mathbf{1}_{A_t}\right]\left(e^{-r_s^T\tau}\varphi_s'(\tau) - e^{-r_s^T\tau}r_s^T\varphi_s(\tau)\right)d\tau \\
&= -\int_t^T \mathbb{E}_s^{F^T}\left[z_\tau \mathbf{1}_{A_t} e^{-r_s^T\tau}\right]\varphi_s'(\tau)d\tau + \int_t^T \mathbb{E}_s^{F^T}\left[z_\tau \mathbf{1}_{A_t} e^{-r_s^T\tau} r_s^T\right]\varphi_s(\tau)d\tau.
\end{aligned}
$$

Consequently, the weak time-derivative of $e^{-r_s^T t} z_t$ in $[s,T]$ is $e^{-r_s^T t}\left(\mathcal{D}_s z_t - r_s^T z_t\right)$.

However this process is null. Therefore, $e^{-r_s^T t} z_t$ has null weak time-derivative in $[s,T]$. Hence, by Proposition A.4, $e^{-r_s^T t} z_t$ is a conditional F^T-martingale and so, for any $t \in [s,T]$ and $\tau \in [t,T]$, we have $\mathbb{E}_t^{F^T}[z_\tau] = e^{r_s^T(\tau-t)} z_t$. When τ goes to T^-, we get that $\mathbb{E}_t^{F^T}[z_\tau]$ tends to $e^{r_s^T(T-t)} z_t$ pointwise.

In addition, z_τ converges to $z_T = 0$ in L_s^1 as τ approaches T^- and so $\mathbb{E}_t^{F^T}[z_\tau]$ tends to zero in L_s^1. By uniqueness of the L_s^1-limit, $z_t = 0$ for all $t \in [s,T]$. This proves uniqueness of the solution of problem (9).

B.2 | Proof of Proposition 3.3

Fix any positive s and consider the limit in probability when T goes to infinity under Assumptions 2.1. By Theorem 3.2 in Qin and Linetsky (2017), $r_s^T \xrightarrow{P} r^\infty$. Since r^∞ is positive, $e^{-r_s^T(T-t)} \xrightarrow{P} 0$ for all $s > 0$ and $t > s$. Therefore, $e^{-r_s^T(T-t)} - e^{-r_t^T(T-t)} \xrightarrow{P} 0$ and so $(\rho_t^T(s,h_T) - \pi_t(h_T))/\mathbb{E}_t^{F^T}[h_T]$ tends to 0.

As for the second convergence, suppose that h_T is strictly positive. Since $r_s^T \xrightarrow{P} r^\infty$ for all $s > 0$ as T goes to infinity, the difference $r_s^T - r_t^T$ converges in probability to zero for all $s > 0$ and $t > s$. In addition, for any positive ε,

$$P\left(\frac{\left|\log \rho_t^T(s,h_T) - \log \pi_t(h_T)\right|}{T-t} > \varepsilon\right) = P\left(\left|r_s^T - r_t^T\right| > \varepsilon\right)$$

and this quantity goes to zero because $r_s^T - r_t^T \xrightarrow{P} 0$ as T increases.

B.3 | Proof of Proposition 3.4

By Lemma A.1 in Appendix A, for any $t > s$

$$\mathbb{E}_s^{F^T}\left[\frac{\rho_t^T(s,h_T)}{\pi_t(h_T)}\right] = e^{r_s^T(t-s)}\mathbb{E}_s^{F^T}\left[\frac{\pi_s(1_T)}{\pi_t(1_T)}\right] = e^{r_s^T(t-s)}\pi_s(1_t) \xrightarrow{P} e^{(r^\infty - r_s^t)(t-s)}$$

as T goes to infinity. As to the second convergence, consider the expression

$$\frac{\rho_t^T(s,h_T)}{\pi_t(h_T)} = e^{r_s^T(t-s)}\frac{\pi_s(1_T)}{\pi_t(1_T)} = e^{r_s^T(t-s)}\frac{\pi_0(1_{T-s})}{\pi_0(1_{T-t})}\frac{\pi_s(1_T)}{\pi_0(1_{T-s})}\frac{\pi_0(1_{T-t})}{\pi_t(1_T)}$$

and recall Assumptions 2.1. Theorem 3.2 in Qin and Linetsky (2017) ensures that, as T goes to infinity,

- $e^{r_s^T(t-s)}$ converges to $e^{r^\infty(t-s)}$ in probability;
- $\pi_0(1_{T-s})/\pi_0(1_{T-s-(t-s)})$ converges to $e^{-r^\infty(t-s)}$ in probability;
- $\pi_s(1_T)/\pi_0(1_{T-s})$ converges to b_s^∞ in the semimartingale topology of Émery (1979);
- $\pi_0(1_{T-t})/\pi_t(1_T)$ converges to $1/b_t^\infty$ in the semimartingale topology.

In the semimartingale topology, the product of convergent processes converges to the product of the respective limit processes. Moreover, the convergence in the semimartingale topology necessarily entails the convergence in probability for any fixed t. Therefore, by Slutsky's theorem, we can conclude that

$$\frac{\rho_t^T(s,h_T)}{\pi_t(h_T)} \xrightarrow{P} \frac{e^{r^\infty(t-s)}}{e^{r^\infty(t-s)}}\frac{b_s^\infty}{b_t^\infty} = \frac{b_s^\infty}{b_t^\infty}, \qquad T \to +\infty.$$

B.4 | Proof of Proposition 3.5

Fix $t \in [s,\tau]$. As we will show in Proposition 4.1, when T goes to infinity, $G_{t,\tau}^T$ converges in probability to $G_{t,\tau}^\infty$. Therefore, $G_{t,\tau}^T h_\tau$ goes to $G_{t,\tau}^\infty h_\tau$ in probability. Since $G_{t,\tau}^T h_\tau$ is also convergent in $L^1(P)$ and this convergence implies the one in probability, by uniqueness of the limit, $G_{t,\tau}^T h_\tau$ tends to $G_{t,\tau}^\infty h_\tau$ in $L^1(P)$. Consequently,

$$\mathbb{E}_t^{F^T}[h_\tau] = \mathbb{E}_t\left[G_{t,\tau}^T h_\tau\right] \xrightarrow{L^1} \mathbb{E}_t\left[G_{t,\tau}^\infty h_\tau\right] = \mathbb{E}_t^{F^\infty}[h_\tau], \qquad T \to +\infty$$

and the convergence is also in probability. In addition, $e^{-r_s^T(\tau-t)} \xrightarrow{P} e^{-r^\infty(\tau-t)}$ and so, by the continuous mapping theorem, when T goes to infinity,

$$\rho_t^T(s,h_\tau) = e^{-r_s^T(\tau-t)}\mathbb{E}_t^{F^T}[h_\tau] \xrightarrow{P} e^{-r^\infty(\tau-t)}\mathbb{E}_t^{F^\infty}[h_\tau] = \rho_t^\infty(s,h_\tau).$$

B.5 | Proof of Proposition 4.1

Assumptions 2.1 hold and we exploit Theorem 3.2 in Qin and Linetsky (2017). Since T goes to infinity, we assume that $T > t+s$ without loss of generality.

As for the first convergence, consider

$$\frac{e^{r_s^T s}\pi_s(1_T)}{\pi_s(1_{T-t})} = e^{r_s^T s}\frac{\pi_0(1_{T-s})}{\pi_0(1_{T-t-s})}\frac{\pi_s(1_T)}{\pi_0(1_{T-s})}\frac{\pi_0(1_{T-t-s})}{\pi_s(1_{T-t})}.$$

When T goes to infinity, we have

- $e^{r_s^T s}$ converges to $e^{r^\infty s}$ in probability;
- $\pi_0(1_{T-s})/\pi_0(1_{T-t-s})$ converges to $e^{-r^\infty t}$ in probability;
- $\pi_s(1_T)/\pi_0(1_{T-s})$ converges to b_s^∞ in the semimartingale topology;
- $\pi_0(1_{T-t-s})/\pi_s(1_{T-t})$ converges to $1/b_s^\infty$ in the semimartingale topology.

As a result, the first convergence of the statement obtains. Similarly,

$$\frac{e^{-r_s^T s}\pi_s(1_{T-t})}{\pi_t(1_T)} = e^{-r_s^T s}\frac{\pi_0(1_{T-t-s})}{\pi_0(1_{T-t})}\frac{\pi_s(1_{T-t})}{\pi_0(1_{T-t-s})}\frac{\pi_0(1_{T-t})}{\pi_t(1_T)}$$

and, when T goes to infinity,

- $e^{-r_s^T s}$ converges to $e^{-r^\infty s}$ in probability;
- $\pi_0(1_{T-t-s})/\pi_0(1_{T-t})$ converges to $e^{r^\infty s}$ in probability;
- $\pi_s(1_{T-t})/\pi_0(1_{T-t-s})$ converges to b_s^∞ in the semimartingale topology;
- $\pi_0(1_{T-t})/\pi_t(1_T)$ converges to $1/b_t^\infty$ in the semimartingale topology.

Consequently, the second convergence of the proposition is established.

In addition, Theorem 3.2 in Qin and Linetsky (2017) ensures that $M_{s,t} = e^{-r^\infty(t-s)}(b_s^\infty/b_t^\infty)G_{s,t}^\infty$. At finite horizons, we have

$$G_{s,t}^T = \left(\frac{e^{r_s^T s}\pi_s(1_T)}{\pi_s(1_{T-t})}\right)^{-1}\left(\frac{e^{-r_s^T s}\pi_s(1_{T-t})}{\pi_t(1_T)}\right)^{-1} M_{s,t}.$$

Since the first factor converges in probability to $e^{r^\infty(t-s)}$ and the second one to b_t^∞/b_s^∞, then $G_{s,t}^T$ converges in probability to $G_{s,t}^\infty$ when T goes to infinity.

B.6 | Proof of Proposition 4.2

From the decomposition of $M_{s,t}$, we have that

$$N_{s,t}^T = e^{(r_s^T - r_t^T)(T-t)}e^{-r^\infty(t-s)}\frac{b_s^\infty}{b_t^\infty}G_{s,t}^\infty.$$

Here, $e^{(r_s^T-r_t^T)(T-t)}$ coincides with the ratio $\pi_t(h_T)/\rho_t^T(s,h_T)$ when an arbitrary payoff h_T is considered. Thus, by Proposition 3.4, it converges in probability to b_t^∞/b_s^∞ as T goes to infinity, ensuring the convergence of $N_{s,t}^T$.

B.7 | Proof of Theorem 4.3

Problem (22). We study the properties of the process defined, at any time t, by $G_s^T e^{r_s^T s}N_{s,t}^T$ that coincides with $e^{-r_s^T t}G_t^T$. Its conditional expectation at time s under the measure P belongs to $L^0(\mathcal{F}_s)$. Moreover, L_s^1-right-continuity at t is due to the fact that

$$\mathbb{E}_s\left[\left|e^{-r_s^T\tau}G_\tau^T - e^{-r_s^T t}G_t^T\right|\right] \leqslant e^{-r_s^T t}\left(\left|e^{-r_s^T(\tau-t)}-1\right|G_s^T + \mathbb{E}_s\left[\left|\mathbb{E}_\tau\left[G_T^T\right]-\mathbb{E}_t\left[G_T^T\right]\right|\right]\right)$$

for all $\tau \geqslant t$. Similarly to the proof of Theorem 3.2, Lévy's downward theorem ensures the convergence to zero when $\tau \to t^+$. A parallel reasoning guarantees L_s^1-left-continuity at T and so $e^{-r_s^T t}G_t^T$ belongs to $\mathcal{U}_s$.

Next, we show that, under the physical measure, the weak time-derivative in $[s,T]$ of $e^{-r_s^T t}G_t^T$ is $-r_s^T e^{-r_s^T t}G_t^T$. By considering any $A_t \in \mathcal{F}_t$ and $\varphi_s \in C_c^1((t,T),L^0(\mathcal{F}_s))$, we have

$$-\int_t^T \mathbb{E}_s\left[e^{-r_s^T\tau}G_\tau^T\mathbf{1}_{A_t}\right]\varphi_s'(\tau)d\tau = -\mathbb{E}_s\left[G_T^T\mathbf{1}_{A_t}\right]\int_t^T e^{-r_s^T\tau}\varphi_s'(\tau)d\tau$$
$$= -\mathbb{E}_s\left[G_T^T\mathbf{1}_{A_t}\right]\int_t^T r_s^T e^{-r_s^T\tau}\varphi_s(\tau)d\tau = \int_t^T \mathbb{E}_s\left[-r_s^T e^{-r_s^T\tau}G_\tau^T\mathbf{1}_{A_t}\right]\varphi_s(\tau)d\tau.$$

Moving back to $N_{s,t}^T$, we showed that $N_{s,t}^T$ belongs to $\mathcal{U}_s^1$ and $\mathcal{D}N_{s,t}^T = -r_s^T N_{s,t}^T$.

Problem (23). It is convenient to consider the process defined, at any t, by $e^{r^\infty s}G_s^\infty N_{s,t}^\infty$, that is $e^{-r^\infty t}G_t^\infty$. Since G_t^∞ is a martingale under P, $\mathbb{E}_s[e^{-r^\infty t}G_t^\infty] = e^{-r^\infty t}G_s^\infty$ belongs to $L^0(\mathcal{F}_s)$. In addition, $\int_s^{+\infty}\mathbb{E}_s[e^{-r^\infty\tau}G_\tau^\infty]d\tau = G_s^\infty e^{-r^\infty s}/r^\infty$ is in $L^0(\mathcal{F}_s)$, too.

L_s^1-right-continuity at any t can be shown as in the proof of problem (22) by observing that $G_\tau^\infty = \mathbb{E}_\tau[G_T^\infty]$ and $G_t^\infty = \mathbb{E}_t[G_T^\infty]$ for any T larger than τ. As a result, $e^{-r^\infty t}G_t^\infty$ is in $\mathcal{U}_s$ with $T = +\infty$.

Regarding weak time-differentiability in $[s, +\infty)$, we can follow again the proof of problem (22) by integrating on intervals $[t, +\infty)$. Indeed, it is enough to use the relation $G_\tau^\infty = \mathbb{E}_\tau[G_T^\infty]$, where T is a time index larger than any instant in the (bounded) supports of φ_s and φ_s'. Thus, the weak time-derivative in $[s, +\infty)$ of $e^{-r^\infty t} G_t^\infty$ is $-r^\infty e^{-r^\infty t} G_t^\infty$. Consequently, $N_{s,t}^\infty$ turns out to be a process in $\mathcal{U}_s^1$ with $T = +\infty$ that satisfies $\mathcal{D} N_{s,t}^\infty = -r^\infty N_{s,t}^\infty$.