Heterogeneous awareness in financial markets*

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Abstract

The overlook of some economic scenarios may result in unforeseen negative outcomes for investors. In this paper, we consider an order-driven financial market in which a fraction of the traders is only partially aware of the possible payoffs of a risky asset, but is aware of the possibility of facing unknown contingencies. Investors decide whether to acquire a costly signal about the payoff of the risky asset and whether to buy such asset given their awareness level and their perceived relations among signals, order flows, and prices. We show that as unawareness becomes more severe, the value of the signal to the partially aware traders diminishes. In turn, through its impact on the price, the reduced number of partially aware informed investors increases the incentives of the fully aware to acquire the signal. In the aggregate, the latter effect does not outweigh the former, so that the overall proportion of informed investors in the market is (weakly) decreasing in the unawareness level. As for the equilibrium price, a lower amount of informed traders makes it more difficult for market makers to distinguish between good and bad signals, and this brings the conditional expectations of the price closer to the unconditional one and reduces the price variance.

JEL classification: D82, D83, G41.
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1

1 Introduction

In most of the literature on information acquisition in financial markets, all agents are assumed to have complete knowledge of the possible risks they are exposed to when trading an asset. Evidence, however, shows that this is often not the case. Gennaioli, Shleifer, and Vishny (2012), for instance, document many episodes in the recent U.S. history where most investors were not aware of important risks associated with several kinds of financial instruments. The most notorious example is provided by the securitization of mortgages during the 2000s. Indeed, as described by Gennaioli et al. (2012), until the summer of 2007 both the holders of mortgage-backed securities and financial intermediaries appeared to be substantially unaware of (i) how fast house prices could decline and mortgage defaults grow, and (ii) the sensitivity of the price of such AAA-rated securities to house prices. In fact, the latter phenomenon was largely overlooked by the models employed by rating agencies (Coval, Jurek, and Stafford, 2009).

Given the past episodes of neglected risks, and in particular the manifest role played by such risks in the global financial crisis, it seems reasonable to consider the idea that, when trading certain assets, investors entertain the possibility of being exposed to unanticipated risks. That is, the investors, being aware of their incomplete knowledge of the risks, may assign a positive probability to the event of facing contingencies different from those they are aware of. It is then interesting to investigate how bounded awareness by part of the market participants impacts agents' incentives to acquire information, and how this in turn affects asset prices.

We propose to tackle these issues using a simple order-driven financial market without short sales *à la* Kyle (1985), where we incorporate the notion of awareness of unawareness as axiomatized in Karni and Vierø (2017). More specifically, in our model agents have the opportunity to buy a risky asset or a risk-free security. Competitive market makers aggregate the orders from rational investors and noise traders and, in a later stage, set the price of the asset equal to its expected payoff conditional on the amount of orders received. Investors can buy a costly signal about the payoff of the risky asset.

A given fraction of the rational traders neglects some risks associated with the risky asset. In particular, they ignore the most negative payoffs of such security. They are, however, aware of their unawareness, in the sense that they assign a positive probability and a utility to the event of incurring outcomes different from those they are aware of. They can use the signal to update the probability of such an event, but the signal does not allow them to discover new outcomes. The market makers and the fully aware agents know the true proportion of the partially aware in the population, while the latter believe that all market participants are partially aware. We impose a certain degree of rationality on the

beliefs of the partially aware agents by requiring that, when updating their beliefs after a signal, they form correct posteriors about the outcomes they are aware of.

We study an equilibrium in which rational investors choose whether to acquire the signal and which asset to buy so as to maximize their expected profit given their level of awareness (and their belief about the proportion of partially aware traders). The erroneous belief of the partially aware that all agents are partially aware leads them to have a distorted view of the relations among signals, order flows, and prices.

We first analyze the equilibrium amounts of informed traders among the fully and the partially aware investors. We show that as unawareness becomes more severe (i.e., as the partially aware traders neglect a larger number of possible asset payoffs), the conditional expected utilities formed by the partially aware investors after the signal realizations become more restricted around the unconditional one. Intuitively, the fact that such agents overlook a larger fraction of possible outcomes weakens their response, in terms of the shift in their expectations, to both good and bad signals. As long as the prior expectations of such agents are not too biased, this reduces the value of the signal to them, and we show that the equilibrium number of partially aware informed traders decreases as their unawareness rises. In turn, such a reduction means that the market makers will be less able to detect good signals, and so they will keep the price relatively low even after these signals. Knowing this, fully aware investors have a greater incentive to get informed, and we show that the number of fully aware informed traders is increasing[1] in the unawareness level of the partially aware. The latter effect, however, does not outweigh the decrease in the amount of the partially aware informed, so that the overall proportion of informed agents in the market is decreasing in the unawareness level.

We then study the impact of heterogeneous awareness on the equilibrium price. Higher unawareness levels imply that it becomes more difficult for market makers to distinguish between good and bad signals since intermediate order flows consistent with both types of signals become more likely. As a consequence, the conditional expectations of the price given the two signals get closer to the unconditional one, i.e. the expected price given a good (resp. bad) signal diminishes (resp. rises) as unawareness becomes more severe. The price variance, on the other hand, decreases since extreme order flows consistent with only one type of signal become less likely.

On the welfare side, when evaluating whether more unawareness is harmful or beneficial, an interesting trade-off emerges. On the one hand, reducing the unawareness level in the market induces more traders to get informed and base their decisions on the signal received, which is clearly beneficial. However, having more informed investors means that market

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[1]Throughout the paper, by *increasing* we mean *weakly increasing* or *non-decreasing*. Similarly for *decreasing*.

makers are more able to distinguish between the two types of signal and this reduces the expected profits of the traders following a good signal.

This paper provides the first model of awareness of unawareness for studying information acquisition in financial markets. It shows how awareness of unawareness reduces traders' incentives to get informed, and how through a price externality this increases the incentives of the fully aware investors to acquire signals about the asset. In addition, it provides clear predictions on how awareness of unawareness by part of the traders impacts the equilibrium price.

The paper takes inspiration from the representation results in Karni and Vierø (2017) to model awareness of unawareness. In Karni and Vierø (2017) the state space is constructed from a set of basic actions and a set of consequences. Agents are assumed to be unaware of some consequences, but aware that their knowledge may be incomplete. The authors provide a subjective expected utility representation of preferences over distributions on acts, where agents assign a positive probability and a utility to the event of facing consequences different from those they are aware of. Another choice theoretic paper featuring awareness of unawareness is Grant and Quiggin (2015). Here the authors augment a standard Savage state space with a set of "surprise" states. In addition, they augment a set of "standard" consequences with two unanticipated consequences, one ranked below the worst possible standard consequence and the other ranked above the best standard consequence. These unanticipated consequences can occur only in surprise states. Agents know that their understanding of the world is incomplete and evaluate acts according to an expected uncertain utility representation. Galanis (2015) shows how, in a multiple state space model, agents aware of all outcomes but unaware of some contingencies (and not aware of their unawareness) may have a negative value of information. In his model, the agents' awareness level is not constant across states, creating a signal that agents can only partially understand. This may in turn lead them to commit information processing errors and behave suboptimally in response to additional signals. By contrast, in our model with constant unawareness of outcomes and awareness of unawareness, information is ex ante always valuable, though the lack of knowledge of some outcomes reduces its value compared to a scenario with full awareness.

As for the financial literature, our notion of equilibrium with unawareness builds upon the one of noisy rational expectations equilibrium mainly developed by Grossman and Stiglitz (1980), Hellwig (1980) and Diamond and Verrecchia (1981). Remarkable contributions can also be found in Gennotte and Leland (1990) and Mele and Sangiorgi (2015), where the interaction between asymmetric information and ambiguity is analyzed.

In terms of the applications of the concept of unawareness to the study of equilibria in

4

financial markets, an early contribution is provided by Kraus and Sagi (2006), where the authors analyze an economy with incomplete markets where investors have non-expected utility preferences with individual taste shocks. They provide the equilibrium conditions in a pure exchange economy with traded securities, where the filtration and the set of traded securities are determined endogenously. In contrast, the model that we propose is based on a Kyle-type market with order flows aggregated by market makers. The expected utility framework is maintained in our case, and our focus is on the impact of unawareness on information acquisition. Asymmetric information is instead absent in Kraus and Sagi (2006), where agents are not differentially informed about the payoff-relevant variables. More recently, Schipper and Zhou (2021) consider an asset market where traders have asymmetric awareness of the components affecting the mean or the variance of the fundamental value of the asset, and interact in a uniform price double auction with exogenous private information. The authors derive the linear symmetric Bayes-Nash equilibrium of the model and study the incentives of the more aware investors to raise the awareness of other traders. Differently from our paper, they do not consider awareness of unawareness, which we argued is an important phenomenon to take into account when studying financial markets, nor do they consider information acquisition. Liu (2017) studies a model with costly information acquisition in a quote-driven asset market with one big professional trader who has superior awareness of the available signals and many small investors who are less aware, and are unaware of their unawareness. He shows that under certain assumptions the loss of information quality is more pronounced for moderate levels of awareness asymmetry. Guerdjikova and Quiggin (2020) consider an infinite-horizon economy where agents can only hold assets yielding payoffs measurable with respect to their awareness. They show, among other things, that such economies can exhibit lack of insurance against idiosyncratic risk and partial insurance against aggregate risk, and that agents with different levels of awareness can survive and influence prices even in the limit. Auster, Kettering, and Kochov (2021) study a dynamic economy where agents' awareness of the possible endowment shocks can increase over time and where traders can reoptimize and purchase insurance. They show how partial awareness can lead investors to experience unintended defaults. The possibility of unexpected defaults induced by unawareness is also studied in Teeple (2022). Finally, Vierø (2021) shows that in a Lucas tree-type model of asset pricing with awareness of unawareness and no information acquisition, the discovery of unforeseen dividend realizations makes asset prices more volatile than under full awareness. This differs from our model where unaware investors do not discover new asset payoffs, and where instead the reduced value of information of such traders leads to a distribution of asset prices more concentrated around its mean.

The remainder of the paper is organized as follows. Section 2 presents the model. Section 3 studies how awareness of unawareness affects information acquisition by fully and partially aware investors. Section 4 analyzes the impact on the equilibrium price. Section 5 provides some illustrative simulations. Section 6 concludes. All the proofs are in the appendix.

2   The model

Consider a unit mass of risk-neutral investors who have the opportunity to buy a risky asset or a risk-free security with null interest rate. The payoff Y of the risky asset can take a finite number of increasing values $y_1, \ldots, y_k$. Before deciding whether to buy the risky asset, traders can become *informed* by acquiring at a cost $c > 0$ a common signal S about Y that can take the realizations s_G or s_B, that we refer to as the good and bad signal, respectively. Both realizations of the signal S have a positive probability and we assume that $P(S = s_B | Y = y)$ is strictly decreasing in y (and hence $P(S = s_G | Y = y)$ is strictly increasing in y). From this it follows that the (conditional and unconditional) expected asset payoffs are ordered as follows:

$$\mathbb{E}\left[Y | S = s_B\right] < \mathbb{E}\left[Y\right] < \mathbb{E}\left[Y | S = s_G\right].$$

Investors are heterogeneous in terms of their awareness of the possible payoffs of the risky asset. Specifically, a given fraction $\bar{\eta}_F \in (0,1)$ of traders is *fully aware*, in the sense that they know all such possible payoffs. By contrast, the remaining part of agents $\bar{\eta}_P = 1 - \bar{\eta}_F$ is *partially aware*, since they know only the values of Y larger than or equal to some $y_{\hat{k}}$, with $\hat{k} \in \{2, \ldots, k\}$. We denote such subset of values of Y by $\mathcal{E}$. They instead do not know the asset values in the subset $\neg\mathcal{E} \equiv \{y_1, \ldots, y_{\hat{k}-1}\}$. However, they recognize that their knowledge is incomplete and therefore assign a positive probability to the event of incurring outcomes different from those they are aware of. They also assign a utility/payoff, that we denote by x, to such an event. In general, there would be no a priori restrictions on the value of x. However, we require that unaware agents are not too biased, in the sense that they realize that what they are missing are outcomes that are worse than the currently known ones, i.e. $x < y_{\hat{k}}$. From the perspective of the partially aware traders, the risky asset payoff is therefore captured by a random variable $\hat{Y}$ that can take the increasing values x, $y_{\hat{k}}, \ldots, y_k$ (see the axiomatization in Karni and Vierø, 2017). We refer to the cardinality of $\neg\mathcal{E}$, that is $\hat{k} - 1$, as the *unawareness level*. Throughout the paper, we will focus on the effect of different unawareness levels while keeping the amount of partially aware traders in the market $\bar{\eta}_P$ fixed.

We make the following assumptions on partial awareness:

(A1) $P(\hat{Y} = y_i) = P(Y = y_i)$ for all $i \geq \hat{k}$;

(A2) $P(S = s|\hat{Y} = y_i) = P(S = s|Y = y_i)$ for all $i \geq \hat{k}$ and all s;

(A3) $P(S = s|\hat{Y} \in \neg\mathcal{E}) = P(S = s|Y \in \neg\mathcal{E})$ for all s.

The first assumption requires that partially aware traders have correct prior beliefs about the payoffs they are aware of, and hence do not overestimate nor underestimate the probability of incurring unknown outcomes. Indeed, such probability is given by

$$P(\hat{Y} \in \neg\mathcal{E}) = 1 - P(\hat{Y} \in \mathcal{E}) = 1 - P(Y \in \mathcal{E}) = P(Y \in \neg\mathcal{E}).$$

The second and third assumptions concern the updating of the partially aware agents. They ensure that when using Bayes' rule $P(\hat{Y} = y_i|S = s) = P(Y = y_i|S = s)$ for all $i \geqslant \hat{k}$ and all realizations s of the signal, and that $P(\hat{Y} \in \neg\mathcal{E}|S = s) = P(Y \in \neg\mathcal{E}|S = s)$. In words, we are requiring that agents form correct posterior beliefs about the outcomes they are aware of. In this way we are imposing a certain degree of rationality on investors' beliefs, meaning that the only mistakes they make when updating come from the individual outcomes they are unaware of.

Regularity assumptions $(A1) - (A3)$ imply that the order of the conditional and unconditional expected payoffs are preserved also under partial awareness. Indeed, it can be shown that

$$\mathbb{E}\left[\hat{Y}|S = s_B\right] < \mathbb{E}\left[\hat{Y}\right] < \mathbb{E}\left[\hat{Y}|S = s_G\right].$$

Rational investors can be divided into four groups: η_{FI} of fully aware informed, η_{PI} of partially aware informed (with $\eta_I \equiv \eta_{FI} + \eta_{PI}$), η_{FU} of fully aware uninformed, and η_{PU} of partially aware uninformed (with $\eta_U \equiv \eta_{FU} + \eta_{PU} = 1 - \eta_I$). Partially aware agents believe that all traders are partially aware, i.e. they assume $\bar{\eta}_P = 1$. Based on this misconception, they maximize their expected utility with respect to the *distorted proportion of informed traders* $\hat{\eta}_I$. In equilibrium, the (true) amount of partially aware informed investors in the population will then reflect this proportion, i.e. $\eta_{PI} = \hat{\eta}_I \bar{\eta}_P$. The fully aware traders instead know the true proportion $\bar{\eta}_P$ of the partially aware in the population and maximize their expected utility with respect to η_{FI} given η_{PI}.

In addition to the rational (fully and partially aware) investors, in the market there are noise traders whose orders are collected by the random variable Z. We assume that Z is independent of Y, $\hat{Y}$ and S, and conditionally independent of Y and $\hat{Y}$ given S. In addition, Z is assumed to be uniformly distributed on the interval $[-\ell/2, \ell/2]$, where $0 < \ell \leqslant 1$ is the *noise level*. Note that the size of the support of noise traders does not exceed the total amount of rational investors, so that prices will not be driven solely by the first type of agents. Negative values of Z can be interpreted as sell orders.

The total order flow T is the sum of the noise Z and the orders of the rational traders. It is collected by market makers who are fully aware and know the true proportion of partially aware investors in the population. As in Kyle (1985), market makers are competitive, they collect the aggregate order flow and, in a *later* stage, set the price $p(T) = \mathbb{E}[Y|T]$.[2] Importantly, however, such price function is known only by the fully aware agents. Indeed, as mentioned above, partially aware traders believe that all investors, as well as the market makers, are partially aware. As a consequence, they consider a *distorted order flow* $\hat{T}$ and a *distorted price function* $\hat{p}(\hat{T}) = \mathbb{E}[\hat{Y}|\hat{T}]$.

Notice that perfect competition among market makers implies that $\mathbb{E}[p] = \mathbb{E}[Y]$ and $\mathbb{E}[\hat{p}] = \mathbb{E}[\hat{Y}]$, so that all uninformed investors are indifferent between buying the risky asset and not doing so. We assume that half of them sets the buy order. Hence, the order flow generated by the uninformed agents (fully plus partially aware) is $\eta_U/2$, where η_U is the total amount of uninformed traders.

We introduce the following notion of equilibrium with heterogeneous awareness.

Definition 1 *Given a fraction $\bar{\eta}_P$ of partially aware investors, an equilibrium with heterogeneous awareness is composed of order flows T and $\hat{T}$, prices p and $\hat{p}$, and proportions of informed traders η_{FI}^*, $\hat{\eta}_I^*$ and η_{PI}^*, such that:*

- *market makers set the price $p = \mathbb{E}[Y|T]$;*

- *given $\hat{\eta}_I^* \in [0, 1]$, partially aware informed (resp. uninformed) traders maximize their conditional (resp. unconditional) expected profit believing that $\bar{\eta}_P = 1$, that the order flow is $\hat{T}$, and that market makers set the distorted price $\hat{p} = \mathbb{E}[\hat{Y}|\hat{T}]$:*

$$\max\left\{\mathbb{E}\left[\hat{Y} - \hat{p}|S\right]; 0\right\} - c, \qquad resp. \qquad \max\left\{\mathbb{E}\left[\hat{Y} - \hat{p}\right]; 0\right\};$$

- *given $\eta_{PI}^* = \hat{\eta}_I^* \bar{\eta}_P$ and $\eta_{FI}^* \in [0, \bar{\eta}_F]$, fully aware informed (resp. uninformed) traders maximize their conditional (resp. unconditional) expected profit, knowing the true $\bar{\eta}_P$ and considering the objective p:*

$$\max\left\{\mathbb{E}\left[Y - p|S\right]; 0\right\} - c, \qquad resp. \qquad \max\left\{\mathbb{E}\left[Y - p\right]; 0\right\};$$

- *given $\hat{\eta}_I^*$, the partially aware informed (resp. uninformed) investors are (resp. are not) willing to get informed:*

$$\begin{cases} \hat{U}_{PI}\left(\hat{\eta}_I^*\right) \leqslant 0 & if \quad \hat{\eta}_I^* = 0, \\ \hat{U}_{PI}\left(\hat{\eta}_I^*\right) = 0 & if \quad \hat{\eta}_I^* \in (0, 1), \\ \hat{U}_{PI}\left(\hat{\eta}_I^*\right) \geqslant 0 & if \quad \hat{\eta}_I^* = 1, \end{cases}$$

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[2]Hence, as in Kyle (1985), market makers do not provide investors with additional information through the price.

where $\hat{U}_{PI}$ is the ex-ante expected utility of the partially aware informed.

- given η_{PI}^* and η_{FI}^*, the fully aware informed (resp. uninformed) investors are (resp. are not) willing to get informed.

$$
\begin{cases}
U_{FI}\left(\eta_{FI}^*\right) \leqslant 0 & if \quad \eta_{FI}^* = 0, \\
U_{FI}\left(\eta_{FI}^*\right) = 0 & if \quad \eta_{FI}^* \in (0, \bar{\eta}_F), \\
U_{FI}\left(\eta_{FI}^*\right) \geqslant 0 & if \quad \eta_{FI}^* = \bar{\eta}_F,
\end{cases}
$$

where U_{FI} is the ex-ante expected utility of the fully aware informed.

3 Information acquisition

In this section, we study how the level of unawareness of the partially aware traders affects investors' incentives to acquire information.

First, we focus on order flows and on the objective and distorted price functions. As we have seen, all uninformed agents are indifferent between buying the risky asset and not doing so, and so half of them sets the order. Noise traders instead set the random order Z. Since the equilibrium price is an average of the conditional expectations of the asset payoff, no informed agent will have an incentive to buy the asset after s_B and all informed traders will purchase it after s_G. The (true) order flow, as correctly perceived by fully aware traders, is therefore

$$
T = Z + \frac{\eta_U}{2} + \begin{cases} 0 & \text{if} \quad S = s_B \\ \eta_I & \text{if} \quad S = s_G \end{cases}
$$

with $0 \leqslant \eta_I \leqslant 1$.

Since partially aware agents believe that all traders are partially aware, they instead consider the distorted order flow

$$
\hat{T} = Z + \frac{\hat{\eta}_U}{2} + \begin{cases} 0 & \text{if} \quad S = s_B \\ \hat{\eta}_I & \text{if} \quad S = s_G. \end{cases}
$$

with $0 \leqslant \hat{\eta}_I \leqslant 1$. Note that the distorted perception of the partially aware can lead to a value of $\hat{\eta}_I$ larger than $\bar{\eta}_P$.

Recalling that the amount of partially aware informed traders is given by $\eta_{PI} = \hat{\eta}_I \bar{\eta}_P$, the overall amount of informed investors is

$$
\eta_I = \eta_{PI} + \eta_{FI} = \hat{\eta}_I \bar{\eta}_P + \eta_{FI}. \tag{1}
$$

The equilibrium value of $\hat{\eta}_I$ is determined by the utility maximization condition of the partially aware (with the constraint $0 \leqslant \hat{\eta}_I \leqslant 1$), while the utility maximization condition of the fully aware determines the equilibrium value of η_{FI} with the constraint $0 \leqslant \eta_{FI} \leqslant \bar{\eta}_F$. Clearly, $\eta_{FI} \leqslant \eta_I$ by eq. (1).
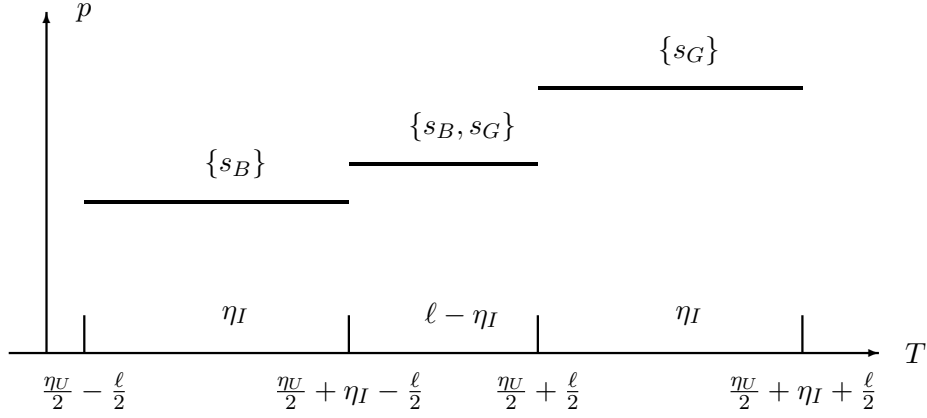
Figure 1: Equilibrium price p as a function of the order flows on their support. Each price level is consistent with the signal realizations written inside the curly brackets. The size of each subinterval depends on the amount of informed traders in equilibrium. An analogous representation holds for the distorted price $\hat{p}$, with $\hat{\eta}_I$, $\hat{\eta}_U$ and $\hat{T}$ instead of η_I, η_U and T.

Turning to prices, since $p = \mathbb{E}[Y|T]$, for any realization t of the order flow T, the equilibrium price satisfies

$$p(t) = \mathbb{E}\left[Y|S = s_G\right] P\left(S = s_G|T = t\right) + \mathbb{E}\left[Y|S = s_B\right] P\left(S = s_B|T = t\right),$$

and an analogous relation holds for the distorted price $\hat{p}$. The true price p and the distorted one $\hat{p}$ are piecewise constant increasing functions of T and $\hat{T}$, respectively. Indeed, the partial overlap of the possible realizations of T due to different signals gives rise to a partition of the support of T into three subintervals, whose width depends on the equilibrium amount of informed traders. The same occurs for $\hat{T}$. For a graphical representation, see Figure 1. Each price level is an average of the (possibly distorted) expected payoffs conditional on the signal realizations written inside the curly brackets. In particular, consider the true price p. If $t \in I_1 \equiv [\eta_U/2 - \ell/2, \eta_U/2 + \eta_I - \ell/2)$, then

$$p(t) = \mathbb{E}\left[Y|S = s_B\right].$$

If $t \in I_2 \equiv [\eta_U/2 + \eta_I - \ell/2, \eta_U/2 + \ell/2)$, then

$$p(t) = \mathbb{E}[Y].$$

Finally, if $t \in I_3 \equiv [\eta_U/2 + \ell/2, \eta_U/2 + \eta_I + \ell/2]$, then

$$p(t) = \mathbb{E}\left[Y|S = s_G\right].$$

10

The size of the regions I_1 and I_3 is η_I, while that of I_2 is $\ell - \eta_I$. Analogous values hold in the case of the distorted price $\hat{p}$, with $\hat{\eta}_I$ and $\hat{\eta}_U$ instead of η_I and η_U. To ensure that the region I_2 is non-empty, we assume that there is enough noise trading in the market such that $\ell \geqslant \eta_I$ and $\ell \geqslant \hat{\eta}_I$. This condition permits to have a region of order flows (I_2) which is consistent with both the good and the bad signal. As a result, the constraints in our equilibrium are

$$0 \leqslant \hat{\eta}_I \leqslant \ell, \qquad 0 \leqslant \eta_{FI} \leqslant \min\left\{\bar{\eta}_F; \ell - \hat{\eta}_I \bar{\eta}_P\right\}.$$

We now focus on how partially aware informed traders form their posterior expected evaluation of the risky asset. After receiving signal s, such agents use Bayes' rule and, by assumptions (A1)-(A3), form correct posterior beliefs about the known payoffs. They instead do not know the single outcomes in $\neg\mathcal{E}$ and thus cannot form beliefs about them. However, the same three assumptions imply that their posterior evaluation of the unknown event is equivalent (from a fully aware perspective) to:

$$P\left(\hat{Y} \in \neg\mathcal{E}|S = s\right) = \frac{P\left(S = s|Y \in \neg\mathcal{E}\right)P\left(Y \in \neg\mathcal{E}\right)}{P(S = s)}$$

$$= \frac{\sum_{i=1}^{\hat{k}-1} P\left(S = s|Y = y_i\right)P\left(Y = y_i\right)}{\sum_{i=1}^{\hat{k}-1} P\left(Y = y_i\right)} \frac{\sum_{i=1}^{\hat{k}-1} P\left(Y = y_i\right)}{P(S = s)}$$

$$= \sum_{i=1}^{\hat{k}-1}\left(\frac{\sum_{j=1}^{\hat{k}-1} P\left(S = s|Y = y_j\right)P\left(Y = y_j\right)}{\sum_{j=1}^{\hat{k}-1} P\left(Y = y_j\right)}\right)\frac{P\left(Y = y_i\right)}{P(S = s)}.$$

From this we see that when updating, the partially aware investors behave as if they were replacing every payoff in $\neg\mathcal{E}$ with x and weighed each of these payoffs using an average of the conditional probabilities of the signal within such set. This averaging, combined with the monotonicity of the conditional probabilities of the signals, implies that, for a given $\mathbb{E}[\hat{Y}]$, the conditional expected utilities of the partially aware traders get more and more restricted around the unconditional one (while preserving their order) as their unawareness level increases. Intuitively, unaware agents cannot distinguish between the outcomes in $\neg\mathcal{E}$, in which higher payoffs are associated with higher (resp. lower) conditional probabilities of the good (resp. bad) signal. They instead update their beliefs as if they were using the same, average conditional probability of the signal for each outcome in $\neg\mathcal{E}$. This averaging reduces their sensitivity to signals in terms of the shifts in posterior expectations, and the larger the set of outcomes over which this averaging takes place, the larger the reduction in their sensitivity to signals.

Formally, we have the following result[3] (see Figure 2 for a graphical representation).

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[3]Throughout the text, when we compare agents with different unawareness levels we will always implicitly

$$\mathbb{E}\left[\hat{Y}_1|S=s_B\right] \qquad\qquad \mathbb{E}\left[\hat{Y}_1\right] \qquad\qquad \mathbb{E}\left[\hat{Y}_1|S=s_G\right]$$

$$\mathbb{E}\left[\hat{Y}_2|S=s_B\right] \qquad\qquad \mathbb{E}\left[\hat{Y}_2\right] \qquad\qquad \mathbb{E}\left[\hat{Y}_2|S=s_G\right]$$
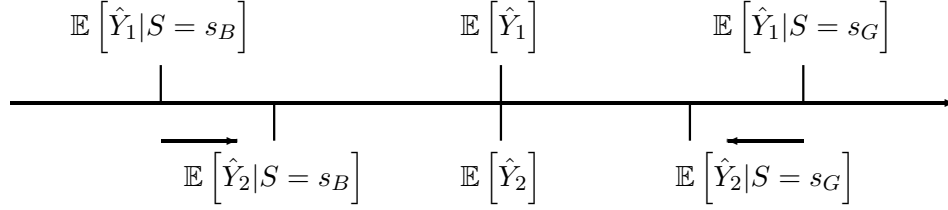
Figure 2: Effect of partial awareness on expected utilities conditional on the different signal realizations. Investor 2 has a greater unawareness level than investor 1.

Proposition 1 *For a given* $\mathbb{E}[\hat{Y}]$, $\mathbb{E}[\hat{Y}|S=s_B]$ *is strictly increasing in the unawareness level, while* $\mathbb{E}[\hat{Y}|S=s_G]$ *is strictly decreasing.*

For a given $\mathbb{E}[\hat{Y}]$, more unaware traders will therefore have more restricted posterior expected utilities. The magnitude of this effect will depend on the values of the additional payoffs the more unaware agents are unaware of and on the corresponding conditional probabilities of the signals. In particular, as can be seen in the proof of Proposition 1, the reduction in the dispersion of the posterior expected utilities will be stronger the higher the values of the additional payoffs and the higher (resp. lower) the conditional probabilities of the good (resp. bad) signal over such payoffs, since in this case the averaging effect described above will have a greater impact on conditional expectations.

Before proceeding with the analysis, it is worth discussing the extent to which our results depend on the fact that investors are unaware of the most negative payoffs of the risky asset. As we have just seen, awareness of unawareness of such payoffs leads to restricted conditional expectations, whilst preserving their order. An analogous result could be obtained if the agents were unaware of the most positive payoffs of the risky asset or, more generally, of all the payoffs within a subinterval of Y, since also in these cases the averaging effect, combined with the monotonicity of the conditional probabilities, would imply a reduced sensitivity to signals, while maintaining the order of the expectations. This would no longer be the case if traders were instead unaware of a generic subset of Y. Here, the averaging of probabilities would cease to have an unambiguous impact on conditional expectations, and may even lead to their reversal, with uncertain effects on unaware agents' incentives to get informed.

The restricted posterior expected utilities induced by awareness of unawareness imply that, if $\mathbb{E}[\hat{Y}] = \mathbb{E}[Y]$, then the partially aware investors will have a lower incentive to

assume that at least one of the additional payoffs the more unaware trader is unaware of has a positive prior probability.

12

buy the signal than the fully aware. This negative impact of awareness of unawareness on information acquisition could however be offset by a sufficiently large difference in their prior evaluation of the asset. Indeed, if partially aware traders are ex ante very biased, meaning that $\mathbb{E}[\hat{Y}]$ differs substantially from $\mathbb{E}[Y]$, the two possibly contrasting effects of more restricted conditional expectations given a shared prior evaluation of the asset on the one hand, and different prior expectations on the other, do not lead to clearcut predictions on agents' incentives to get informed. In what follows, we therefore focus on the case where partially aware investors are not ex ante too biased, in the sense that $\mathbb{E}[\hat{Y}]$ does not differ too much from $\mathbb{E}[Y]$ or, equivalently, x is sufficiently close to the expected value of the asset payoff over the set of unknown outcomes $\neg\mathcal{E}$ (see Lemma 1 below).

We now show that, as long as partially aware traders are not ex ante too biased, five equilibrium configurations can arise, including corner solutions. In particular, in the next two propositions, we use the signal cost c as the discriminating parameter for the different equilibria. More specifically, the threshold cost $\hat{\gamma}$, defined by

$$\hat{\gamma} \equiv \left(\mathbb{E}\left[\hat{Y}|S = s_G\right] - \mathbb{E}\left[\hat{Y}\right]\right) P\left(S = s_G\right),$$

allows us to distinguish between corner solutions (Proposition 2) and internal solutions (Proposition 3). This cost depends on the unawareness level. The fully aware counterpart of $\hat{\gamma}$ is the parameter γ defined by

$$\gamma \equiv \left(\mathbb{E}\left[Y|S = s_G\right] - \mathbb{E}[Y]\right) P\left(S = s_G\right).$$

Before describing the equilibria, we show that when partially aware investors are not ex ante too biased, then $\hat{\gamma} \leqslant \gamma$.

Lemma 1 *If*

$$|x - \mathbb{E}\left[Y|Y \in \neg\mathcal{E}\right]| \leqslant \frac{\sum_{i=1}^{\hat{k}-1} P\left(Y = y_i|S = s_G\right)\left(y_i - \mathbb{E}\left[Y|Y \in \neg\mathcal{E}\right]\right)}{\left|\sum_{i=1}^{\hat{k}-1}\left(P\left(Y = y_i|S = s_G\right) - P\left(Y = y_i\right)\right)\right|}, \tag{2}$$

then $\hat{\gamma} \leqslant \gamma$.

The following proposition considers the case where the signal cost is high enough so that no partially aware trader has an incentive to get informed.

Proposition 2 *If $c \geqslant \hat{\gamma}$ and condition (2) holds, three equilibria are possible.*

1. If $c \geqslant \gamma$, then $\eta_{PI}^* = \eta_{FI}^* = \eta_I^* = 0$.

2. If $(\ell - \bar{\eta}_F)\gamma/\ell \leqslant c < \gamma$, then $\eta_{PI}^* = 0$ and $\eta_{FI}^* = \eta_I^* = \ell(1 - c/\gamma)$.

3. If $c < (\ell - \bar{\eta}_F)\gamma/\ell$, then $\eta_{PI}^* = 0$ and $\eta_{FI}^* = \eta_I^* = \bar{\eta}_F$.

As shown in Proposition 1, awareness of unawareness leads to more restricted posterior expected utilities which, as long as the partially aware traders are not ex ante too biased, makes the signal less valuable to them. Indeed, as Proposition 2 shows, there exists an interval of "intermediate" signal costs, $c \in [\hat{\gamma}, \gamma)$, for which only the fully aware investors get informed. Note that in this case the amount of informed traders in the market is unaffected by the unawareness level of the partially aware. This occurs because, since none of these investors gets informed, they do not affect the price level and hence the incentives of the fully aware to acquire the signal.

The next proposition instead considers the case in which the signal cost is low enough so that both types of investor have an incentive to get informed.

Proposition 3 *If $c < \hat{\gamma}$ and condition (2) holds, two equilibria are possible.*

4. *If*

$$\bar{\eta}_F \geqslant \frac{\ell c \,(\gamma - \hat{\gamma})}{\gamma \,(\ell c + (1 - \ell)\hat{\gamma})}, \tag{3}$$

then

$$\eta_{PI}^* = \ell\left(1 - \frac{c}{\hat{\gamma}}\right)\bar{\eta}_P, \qquad \eta_{FI}^* = \ell\left(\bar{\eta}_F - c\left(\frac{1}{\gamma} - \frac{\bar{\eta}_P}{\hat{\gamma}}\right)\right) \qquad and \qquad \eta_I^* = \ell\left(1 - \frac{c}{\gamma}\right).$$

5. *Otherwise,*

$$\eta_{PI}^* = \ell\left(1 - \frac{c}{\hat{\gamma}}\right)\bar{\eta}_P, \qquad \eta_{FI}^* = \bar{\eta}_F \qquad and \qquad \eta_I^* = \bar{\eta}_P\ell\left(1 - \frac{c}{\hat{\gamma}}\right) + \bar{\eta}_F.$$

In the fourth case, η_{PI}^* is decreasing in the unawareness level (through $\hat{\gamma}$), η_{FI}^* is increasing, while η_I^* is constant. As unawareness rises, fewer partially aware investors are willing to get informed since their more restricted posterior expected utilities make the signal less valuable to them. This in turn increases the incentives of the fully aware to purchase the signal because, *ceteris paribus*, a lower amount of partially aware informed traders makes it harder for market makers to distinguish between the two signals, which keeps the expected price relatively low even after a good signal. The two effects compensate each other in a way that the overall amount of informed traders in the market remains unchanged.

The fifth case brings the above logic to the extreme. All fully aware investors get informed, and η_{FI}^* is no longer able to compensate the decrease in η_{PI}^*. As a result, the total amount of informed investors is decreasing in the unawareness level. Note that the threshold value for $\bar{\eta}_F$ in (3) is increasing in the unawareness level. Intuitively, the more severe the unawareness of the partially aware, the greater the incentives of the fully aware

14

to buy the signal, and the more likely an equilibrium where all of them are informed. Also note that in both these equilibria, in line with the literature on noisy rational expectations equilibria (e.g. Grossman and Stiglitz, 1980), the overall amount of informed agents in the market is increasing in the noise level ℓ.

Both the third and the fifth case share $\eta_{FI}^* = \bar{\eta}_F$, meaning that all fully aware traders get informed. One can also see the third case as a degeneration of the fifth one where η_{PI}^* is null. Indeed, when the unawareness level rises, fewer and fewer partially aware traders will buy the costly signal in the fifth case. At some point, none of them will get informed and the equilibrium of the third case will arise.

Three main messages emerge from the above discussion: (i) more severe unawareness decreases the incentives of the partially aware traders to get informed, (ii) through its impact on the price, this increases the incentives of the fully aware to acquire the signal, and (iii) in the aggregate, the second effect does not outweigh the first one, so that the overall amount of informed agents in the market is (weakly) decreasing in the unawareness level. Formally, the dependence of the equilibrium amounts of informed investors from the unawareness level can be summarized in the following corollary to Propositions 2 and 3.

Corollary 1 *For a given $\mathbb{E}[\hat{Y}]$, in equilibrium, η_I^* and η_{PI}^* are decreasing in the unawareness level, while η_{FI}^* is increasing.*

4 Equilibrium price

A consequence of Corollary 1 is that the order-flow regions in Figure 1 change their size depending on the unawareness level. Specifically, when such level increases, the extreme regions I_1 and I_3, where order flows are consistent with only one signal realization, shrink. By contrast, the middle region I_2, in which order flows are consistent with both signals, widens. As a result, it becomes more difficult for market makers to distinguish between the two signals, and this leads to a decrease (resp. increase) in the conditional expectation of the price given the good (resp. bad) signal. Formally, we have the following result.

Corollary 2 *For a given $\mathbb{E}[\hat{Y}]$, in equilibrium, $\mathbb{E}[p|S = s_G]$ is decreasing in the unawareness level, while $\mathbb{E}[p|S = s_B]$ is increasing.*

An analogous result holds for the conditional expectations of the distorted price $\hat{p}$. In this case, the two conditional expectations get closer to the unconditional one as unawareness rises not only because of the decrease in $\hat{\eta}_I^*$, but also because of the more restricted conditional expected asset payoffs (see Proposition 1).

As explained above, the reduction in the amount of informed traders due to higher unawareness levels implies that less extreme order flows become more likely. Another consequence of this is that the variance of the equilibrium price decreases as unawareness becomes more severe.

Corollary 3 *For a given* $\mathbb{E}[\hat{Y}]$, *in equilibrium,* $var(p)$ *is decreasing in the unawareness level.*

A similar reasoning shows that the equilibrium price kurtosis $kurt(p) = \mathbb{E}[(p-\mathbb{E}[p])^4]/var(p)^2$ is increasing in the unawareness level.

We now spend some words on the ex-post utility of informed traders and the related welfare analysis. Even though the incentives to get informed are misaligned between fully and partially aware investors, both types of agents share the same ex-post utility V_I in equilibrium (recall that the distorted price $\hat{p}$ does not realize in the market):

$$V_I = \begin{cases} -c & \text{if } S = s_B \\ \mathbb{E}\left[Y|S = s_G\right] - \mathbb{E}\left[p|S = s_G\right] - c & \text{if } S = s_G. \end{cases}$$

By Corollary 2, in equilibrium the ex-post utility of each informed investor is increasing in the unawareness level. By contrast, the aggregate ex-ante expected utility of traders,

$$\left(\left(\mathbb{E}\left[Y|S = s_G\right] - \mathbb{E}\left[p|S = s_G\right]\right) P(S = s_G) - c\right) \eta_I^*,$$

has a quadratic dependence on η_I^*. Here a trade-off emerges. On the one hand, reducing the unawareness in the market would induce more traders to get informed and base their decisions on the signal received, which is clearly beneficial to them. On the other hand, having more informed investors means that market makers are more able to distinguish between the two signals and, as a result, the expected profits of the traders following a good signal would be reduced. Which of the two effects prevails depends on the model's parameters.

As for the "surprise" faced by the partially aware, i.e. the difference between the actual and their distorted perception of expected profits, this is given by

$$\left(\mathbb{E}\left[Y|S = s_G\right] - \mathbb{E}\left[p|S = s_G\right] - \left(\mathbb{E}\left[\hat{Y}|S = s_G\right] - \mathbb{E}\left[\hat{p}|S = s_G\right]\right)\right) P\left(S = s_G\right), \quad (4)$$

where a positive value corresponds to an unexpected gain for such traders, whereas a negative value corresponds to an unanticipated loss. Both of these two cases are in fact possible. In particular, by substituting the expressions of p and $\hat{p}$, it can be easily shown that (4) is negative when

$$\left(\mathbb{E}\left[Y|S = s_G\right] - \mathbb{E}\left[Y\right]\right)\left(\ell - \eta_I\right) \leqslant \left(\mathbb{E}\left[\hat{Y}|S = s_G\right] - \mathbb{E}\left[\hat{Y}\right]\right)\left(\ell - \hat{\eta}_I\right).$$

Here again two contrasting effects of awareness of unawareness are at work. On the one hand, agents that are aware of being unaware (and are not ex ante too biased) underestimate the value of the asset after a good signal, and this points toward an unexpected gain after such signal is realized. On the other hand, these traders also underestimate the overall number of informed investors in the market, which after a good signal leads to prices higher than anticipated, and this points toward an unexpected loss. Each of these two effects can prevail in equilibrium depending on the parameters.

It is worth noting that the signal S per se does not provide partially aware traders with any additional information about what payoffs are contained in $\neg\mathcal{E}$. However, it allows them to better assess the likelihood of incurring unknown outcomes. It is reasonable to expect that if partially aware investors could use a costly "outcome verification technology", such as hiring a consultant, to investigate into $\neg\mathcal{E}$, then they would be more willing to do so when the likelihood of such an event is sufficiently high. In this respect, by affecting traders' incentives to update their estimates of the likelihood of facing unknown contingencies, the unawareness level could also impact their ability to discover new possible outcomes. This is something we wish to investigate in future research.

Finally, one may wonder what happens to information acquisition if partially aware traders are absent in the market, i.e. if $\bar{\eta}_P = 0$ and $\bar{\eta}_F = 1$. In this case, the overall amount of informed agents in the market is higher, since the negative impact of unawareness on information acquisition plays no role. In particular, from the profit maximization of the fully aware we have that only two cases are possible. If $c \geqslant \gamma$, then $\eta^*_{FI} = \eta^*_I = 0$, while if $c < \gamma$ we have the internal solution $\eta^*_{FI} = \eta^*_I = \ell(1 - c/\gamma)$. Note that fully aware investors will never have an incentive to get all informed, since in this case market makers would be perfectly able to distinguish between the two signals and would set the price equal to the conditional expected asset payoff, leading to negative expected profits for the informed traders.

5   Simulations

We now provide some illustrative simulations of the number of informed traders and the expected prices arising in equilibrium. We directly introduce a positive parameter ε and set $\mathbb{E}[\hat{Y}|S = s_B] = \mathbb{E}[Y|S = s_B] + \varepsilon$. An increase in ε corresponds to an increase in the unawareness level.

We first set $\ell = 0.8$, $\bar{\eta}_F = 0.4$, $P(S = s_B) = 0.4$, $c = 0.7$, $\mathbb{E}[Y|S = s_B] = 1$, $\mathbb{E}[Y|S = s_G] = 10$, $E[\hat{Y}] = E[Y] + 0.01$ and we let ε vary between 0.01 and $\mathbb{E}[Y] - \mathbb{E}[Y|S = s_B]$. In the left panel of Figure 3 we represent the proportion of informed traders, η_I, as well as the proportions of the fully and the partially aware informed, η_{FI} and η_{PI},

in equilibrium for increasing values of the unawareness parameter ε. We observe three types of equilibria for low, intermediate, and high values of ε. When the unawareness level is low, we are in the fourth type of equilibrium of Proposition 3: the compensation between the increasing number of the fully aware informed and the decreasing number of the partially aware informed keeps the overall amount of informed traders in the market constant. For intermediate values of ε, the fifth equilibrium configuration emerges: η_{FI} stabilizes at $\bar{\eta}_F$ while the amount of partially aware informed traders keeps falling until it reaches zero. The overall proportion of informed investors inherits this decreasing behavior. Finally, high values of the unawareness parameter lead to the third equilibrium configuration where no partially aware agent gets informed and all fully aware traders buy the signal: $\eta_I = \eta_{FI} = \bar{\eta}_F$.

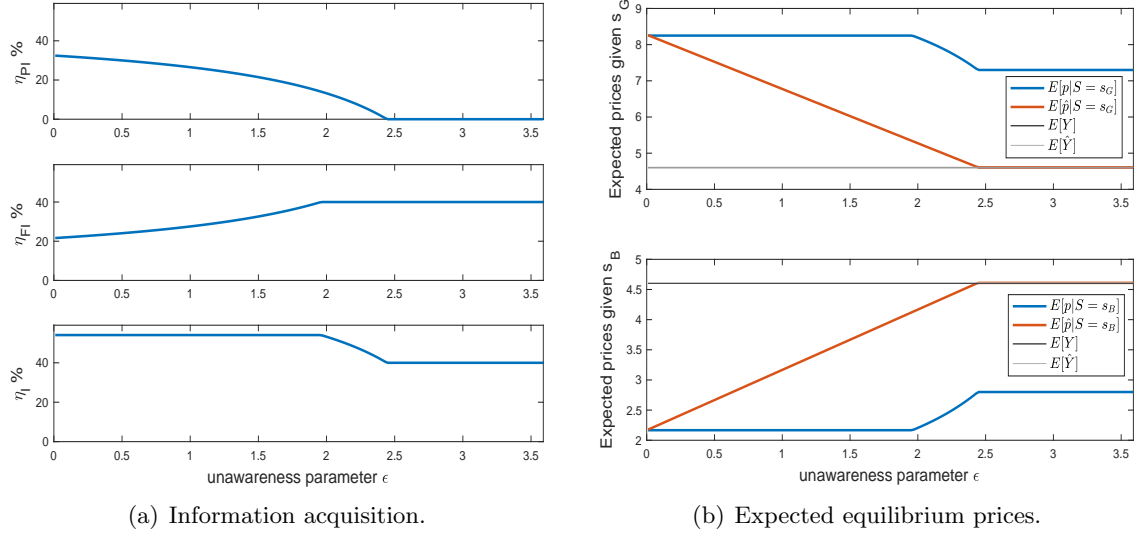(a) Information acquisition.　　　　　　　　(b) Expected equilibrium prices.

Figure 3: Information acquisition and equilibrium prices with heterogeneous awareness ($\ell = 0.8$). Left panel: proportions of informed, fully aware informed, and partially aware informed traders as a function of the unawareness level. Right panel: expected equilibrium price p and expected distorted price $\hat{p}$ after the good and the bad signal as a function of the unawareness level.

As for the prices, the lower amount of informed traders due to higher unawareness levels makes it harder for market makers to distinguish between the two signals. As a result, given a good signal realization the expectations of both the equilibrium and the distorted price decrease as unawareness rises. The opposite behavior characterizes the expected prices after a bad signal.

In Figure 4 we repeat the same simulation using a lower noise parameter value $\ell = 0.3$ and keeping the other parameters unchanged. In this case, as unawareness increases we

18

move from the fourth to the second equilibrium. Intuitively, the lower amount of noise traders makes it easier for market makers to distinguish between the two signals, which reduces the incentives of the fully aware to get informed, so that now η_{FI} always remains strictly below $\bar{\eta}_F$. As unawareness grows, the decrease in the number of the partially aware informed is perfectly compensated by the rise in the amount of the fully aware informed, and the overall proportion of informed investors remains the same across the two equilibria. As a consequence of this, the conditional expectations of the equilibrium price, which are based on η_I, remain constant. By contrast, the conditional expectations of the distorted price change in a way similar to that in the previous simulation. This occurs because such expectations involve $\hat{\eta}_I$ and the distorted expectations of the asset payoff, and both of these quantities change with the unawareness level.

(a) Information acquisition. (b) Expected equilibrium prices.
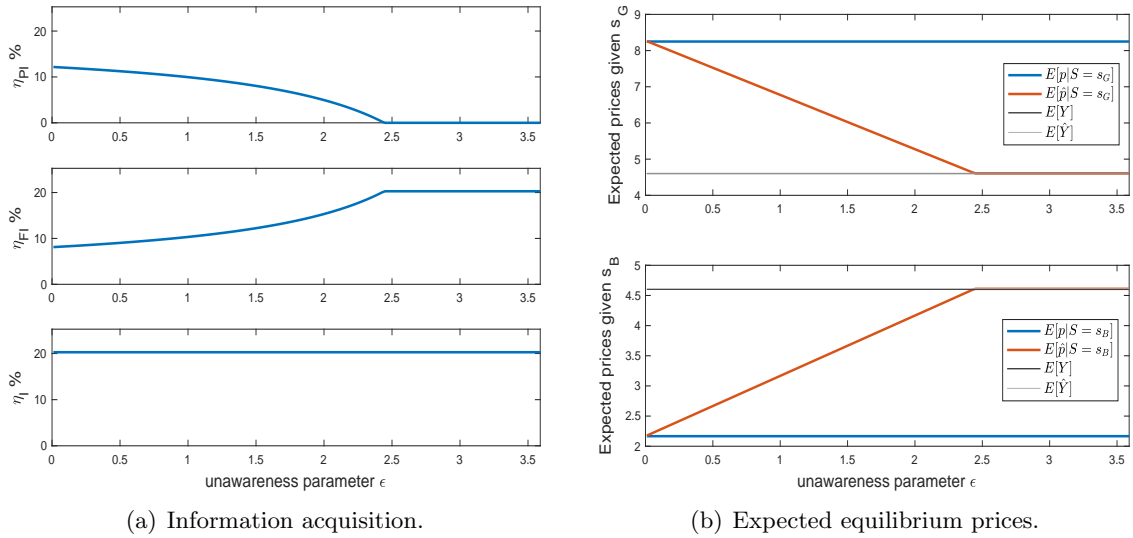
Figure 4: Information acquisition and equilibrium prices with heterogeneous awareness ($\ell = 0.3$). Left panel: proportions of informed, fully aware informed, and partially aware informed traders as a function of the unawareness level. Right panel: expected equilibrium price p and expected distorted price $\hat{p}$ after the good and the bad signal as a function of the unawareness level.

6   Conclusion

Evidence shows that there have been various episodes in the recent U.S. history in which many investors were not aware of important risks associated with several types of financial instruments. Given these past episodes of neglected risks, and in particular the manifest role of such risks in the global financial crisis, it seems reasonable that, when trading certain

assets, investors entertain the possibility of being exposed to unknown risks. We study an order-driven financial market in which some of the traders ignore the most negative payoffs of a risky asset, but assign a positive probability to the event of facing contingencies different from those they are aware of. Such partially aware investors believe that all agents in the market are partially aware and, as a consequence, have a distorted view of the relations among signals, order flows, and asset prices. We analyze an equilibrium in which rational traders decide whether to acquire a costly signal and whether to buy the asset given their level of awareness.

We show that when the partially aware agents overlook a larger fraction of possible asset payoffs their response, in terms of the shift in their expectations, to both good and bad signals gets weaker. As long as the prior expectations of such agents are not too biased, this reduces their perceived value of the signal, leading to a decrease in the number of partially aware informed traders. This, in turn, increases the incentives of the fully aware to get informed to exploit the reduced sensitivity of the price to different signal realizations. Specifically, we show that the number of fully aware informed investors is increasing in the unawareness level of the partially aware. In the aggregate, the latter effect does not outweigh the former, so that the overall amount of informed agents in the market is decreasing in the unawareness level.

As unawareness becomes more severe, the reduced number of informed traders makes it more difficult for market makers to distinguish between good and bad signals, which brings the conditional expectations of the equilibrium price closer to the unconditional one and reduces the price variance. On the welfare side, more unawareness reduces the number of traders that are willing to buy the signal, but increases the actual gains that can be obtained after such signal is realized.

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Appendix

Proof of Proposition 1

Without loss of generality, consider $\neg\mathcal{E}_1 \equiv \{y_1, y_2, \ldots, y_{\hat{k}-1}\}$ and $\neg\mathcal{E}_2 \equiv \neg\mathcal{E}_1 \cup \{y_{\hat{k}}\}$ and, accordingly, the variables $\hat{Y}_1$ and $\hat{Y}_2$ that take values x_1 and x_2 in the event an unknown outcome occurs. Since $\mathbb{E}[\hat{Y}_2] = \mathbb{E}[\hat{Y}_1]$, the values x_1 and x_2 satisfy the relation

$$\sum_{i=\hat{k}+1}^{k} P\left(\hat{Y}_2 = y_i\right) y_i + P\left(\hat{Y}_2 \in \neg\mathcal{E}_2\right) x_2$$
$$= \sum_{i=\hat{k}+1}^{k} P\left(\hat{Y}_1 = y_i\right) y_i + P\left(\hat{Y}_1 = y_{\hat{k}}\right) y_{\hat{k}} + P\left(\hat{Y}_1 \in \neg\mathcal{E}_1\right) x_1.$$

Recall that by assumption (A1) the prior probabilities of $\hat{Y}_1$ and $\hat{Y}_2$ coincide with those of Y. After some simple rearrangements we get

$$x_2 = \frac{P\left(Y \in \neg\mathcal{E}_1\right)}{P\left(Y \in \neg\mathcal{E}_2\right)} x_1 + \frac{P\left(Y = y_{\hat{k}}\right)}{P\left(Y \in \neg\mathcal{E}_2\right)} y_{\hat{k}}. \tag{5}$$

Since the proof of the inequality with s_B is analogous, we only show that $\mathbb{E}[\hat{Y}_2|S = s_G] < \mathbb{E}[\hat{Y}_1|S = s_G]$, that is

$$\sum_{i=\hat{k}+1}^{k} P\left(\hat{Y}_2 = y_i|S = s_G\right) y_i + P\left(\hat{Y}_2 \in \neg\mathcal{E}_2|S = s_G\right) x_2$$
$$< \sum_{i=\hat{k}+1}^{k} P\left(\hat{Y}_1 = y_i|S = s_G\right) y_i + P\left(\hat{Y}_1 = y_{\hat{k}}|S = s_G\right) y_{\hat{k}} + P\left(\hat{Y}_1 \in \neg\mathcal{E}_1|S = s_G\right) x_1.$$

By assumptions (A1)-(A3), the posterior probabilities of $\hat{Y}_1$ and $\hat{Y}_2$ over the known payoffs and the set of unknown outcomes coincide with those of Y. We therefore get the equivalent relation

$$P(Y \in \neg\mathcal{E}_2|S = s_G)x_2 < P(Y = y_{\hat{k}}|S = s_G)y_{\hat{k}} + P(Y \in \neg\mathcal{E}_1|S = s_G)x_1.$$

We now apply Bayes' rule, use eq. (5) and multiply by $P(S = s_G)$:

$$\sum_{i=1}^{\hat{k}} P\left(S = s_G|Y = y_i\right) P\left(Y = y_i\right) \left(\frac{P\left(Y \in \neg\mathcal{E}_1\right)}{P\left(Y \in \neg\mathcal{E}_2\right)} x_1 + \frac{P\left(Y = y_{\hat{k}}\right)}{P\left(Y \in \neg\mathcal{E}_2\right)} y_{\hat{k}}\right)$$
$$< \sum_{i=1}^{\hat{k}-1} P\left(S = s_G|Y = y_i\right) P\left(Y = y_i\right) x_1 + P\left(S = s_G|Y = y_{\hat{k}}\right) P\left(Y = y_{\hat{k}}\right) y_{\hat{k}}.$$

We multiply both sides by $P(Y \in \neg\mathcal{E}_2)$ and rearrange the terms:

$$\sum_{i=1}^{\hat{k}-1} P\left(S = s_G | Y = y_i\right) P\left(Y = y_i\right) \left(P\left(Y \in \neg\mathcal{E}_1\right) - P\left(Y \in \neg\mathcal{E}_2\right)\right) x_1$$
$$+ P\left(S = s_G | Y = y_{\hat{k}}\right) P\left(Y = y_{\hat{k}}\right) P\left(Y \in \neg\mathcal{E}_1\right) x_1$$
$$< -\sum_{i=1}^{\hat{k}-1} P\left(S = s_G | Y = y_i\right) P\left(Y = y_i\right) P\left(Y = y_{\hat{k}}\right) y_{\hat{k}}$$
$$+ P\left(S = s_G | Y = y_{\hat{k}}\right) P\left(Y = y_{\hat{k}}\right) \left(P\left(Y \in \neg\mathcal{E}_2\right) - P\left(Y = y_{\hat{k}}\right)\right) y_{\hat{k}}.$$

We now use the relation $P(Y \in \neg\mathcal{E}_2) = P(Y \in \neg\mathcal{E}_1) + P(Y = y_{\hat{k}})$ and divide both sides of the inequality by $P(Y = y_{\hat{k}})$:

$$\left[-\sum_{i=1}^{\hat{k}-1} P\left(S = s_G | Y = y_i\right) P\left(Y = y_i\right) + P\left(S = s_G | Y = y_{\hat{k}}\right) P\left(Y \in \neg\mathcal{E}_1\right) \right] \left(x_1 - y_{\hat{k}}\right) < 0.$$

This inequality is satisfied because $x_1 < y_{\hat{k}}$ and the expression in square brackets is positive. Indeed, $P(S = s_G | Y = y_{\hat{k}}) > P(S = s_G | Y = y_i)$ for all $i = 1, \dots, \hat{k} - 1$ thanks to the monotonicity of $P(S = s_G | Y = y_i)$ in y_i.

Once we have proved the result for $\neg\mathcal{E}_1$ versus $\neg\mathcal{E}_2$, we can do the same for $\neg\mathcal{E}_2$ versus $\neg\mathcal{E}_3$, where $\neg\mathcal{E}_3$ contains an additional payoff compared to $\neg\mathcal{E}_2$. Then, by the transitive property, the result of the proposition holds also for $\neg\mathcal{E}_1$ versus $\neg\mathcal{E}_3$, and so on.

Proof of Lemma 1

To prove that $\hat{\gamma} \leqslant \gamma$, we show that

$$\mathbb{E}\left[\hat{Y}|S = s_G\right] - \mathbb{E}\left[Y|S = s_G\right] \leqslant \mathbb{E}\left[\hat{Y}\right] - \mathbb{E}[Y].$$

Equivalently,

$$xP\left(\hat{Y} \in \neg\mathcal{E}|S = s_G\right) + \sum_{i=\hat{k}}^{k} y_i P\left(\hat{Y} = y_i | S = s_G\right)$$
$$- \sum_{i=1}^{\hat{k}-1} y_i P\left(Y = y_i | S = s_G\right) - \sum_{i=\hat{k}}^{k} y_i P\left(Y = y_i | S = s_G\right)$$
$$\leqslant xP\left(\hat{Y} \in \neg\mathcal{E}\right) + \sum_{i=\hat{k}}^{k} y_i P\left(\hat{Y} = y_i\right) - \sum_{i=1}^{\hat{k}-1} y_i P\left(Y = y_i\right) - \sum_{i=\hat{k}}^{k} y_i P\left(Y = y_i\right).$$

23

By assumptions (A1)-(A3), all $\hat{Y}$ can be replaced by Y above. Hence, after simplifying,

$$xP\left(Y \in \neg\mathcal{E}|S = s_G\right) - \sum_{i=1}^{\hat{k}-1} y_i P\left(Y = y_i|S = s_G\right) \leqslant xP\left(Y \in \neg\mathcal{E}\right) - \sum_{i=1}^{\hat{k}-1} y_i P\left(Y = y_i\right).$$

Since $P(Y \in \neg\mathcal{E}) = \sum_{i=1}^{\hat{k}-1} P(Y = y_i)$ and an analogous relation holds for the conditional probabilities, we obtain

$$\sum_{i=1}^{\hat{k}-1} P\left(Y = y_i|S = s_G\right)(x - y_i) \leqslant \sum_{i=1}^{\hat{k}-1} P\left(Y = y_i\right)(x - y_i).$$

We now add and subtract $\mathbb{E}[Y|Y \in \neg\mathcal{E}]$ as follows:

$$\sum_{i=1}^{\hat{k}-1} P\left(Y = y_i|S = s_G\right)(x - \mathbb{E}\left[Y|Y \in \neg\mathcal{E}\right]) + \sum_{i=1}^{\hat{k}-1} P\left(Y = y_i|S = s_G\right)(\mathbb{E}\left[Y|Y \in \neg\mathcal{E}\right] - y_i)$$

$$\leqslant \sum_{i=1}^{\hat{k}-1} P\left(Y = y_i\right)(x - \mathbb{E}\left[Y|Y \in \neg\mathcal{E}\right]) + \sum_{i=1}^{\hat{k}-1} P\left(Y = y_i\right)(\mathbb{E}\left[Y|Y \in \neg\mathcal{E}\right] - y_i).$$

Since $\sum_{i=1}^{\hat{k}-1} P(Y = y_i)(\mathbb{E}[Y|Y \in \neg\mathcal{E}] - y_i) = 0$, we obtain

$$\sum_{i=1}^{\hat{k}-1} \left(P\left(Y = y_i|S = s_G\right) - P\left(Y = y_i\right)\right)(x - \mathbb{E}\left[Y|Y \in \neg\mathcal{E}\right])$$

$$\leqslant \sum_{i=1}^{\hat{k}-1} P\left(Y = y_i|S = s_G\right)(y_i - \mathbb{E}\left[Y|Y \in \neg\mathcal{E}\right]),$$

where the last term is positive by the increasing monotonicity of $P(S = s_G|Y = y_i)$ in y_i. This inequality is satisfied when the condition in eq. (2) holds.

Proof of Proposition 2

To ease the notation, we denote by $[w]$ the interval $[w - \ell/2, w + \ell/2]$ for any real number w. First, we focus on the conditional probabilities of the order flow T given the signal. These are given by

$$\wp_{T|S=s_G}(t) = \frac{1}{\ell} 1_{[\eta_U/2+\eta_I]}(t), \qquad \wp_{T|S=s_B}(t) = \frac{1}{\ell} 1_{[\eta_U/2]}(t),$$

where 1 denotes the indicator function. Similarly, the unconditional probabilities are given by

$$\wp_T(t) = \frac{1}{\ell} \left(1_{[\eta_U/2+\eta_I]}(t)P\left(S = s_G\right) + 1_{[\eta_U/2]}(t)P\left(S = s_B\right)\right). \tag{6}$$

Moreover, we have

$$P\left(S = s_G | T = t\right) = \frac{P(S = s_G) 1_{\left[\frac{\eta_U}{2} + \eta_I\right]}(t)}{1_{\left[\frac{\eta_U}{2} + \eta_I\right]}(t) P\left(S = s_G\right) + 1_{\left[\frac{\eta_U}{2}\right]}(t) P\left(S = s_B\right)}$$

and

$$P\left(S = s_B | T = t\right) = \frac{P(S = s_B) 1_{\left[\frac{\eta_U}{2}\right]}(t)}{1_{\left[\frac{\eta_U}{2} + \eta_I\right]}(t) P\left(S = s_G\right) + 1_{\left[\frac{\eta_U}{2}\right]}(t) P\left(S = s_B\right)}.$$

Similar results hold for the distorted order flow $\hat{T}$ and the distorted price $\hat{p}$.

To make explicit the dependence of the utility functions on the amount of informed traders, we first make explicit the expression of $\mathbb{E}[p|S = s_G]$. Such expected price is an average of the two conditional expectations of the asset payoff. This occurs because, depending on the amount of noise trading in the market, s_G can lead to order flows that are consistent with both the good and the bad signal. In particular, it can be easily shown that

$$\mathbb{E}\left[p | S = s_G\right] = \int_{\text{supp} T | S = s_G} \left(\sum_{\sigma = s_B, s_G} \mathbb{E}\left[Y | S = \sigma\right] P\left(S = \sigma | T = t\right) \wp_{T | S = s_G}(t) \right) dt.$$

From the above-written conditional probabilities, we get

$$\mathbb{E}\left[p | S = s_G\right] = \int_{\text{supp} T | S = s_G} \left(\mathbb{E}\left[Y | S = s_G\right] P\left(S = s_G | T = t\right) \wp_{T | S = s_G}(t) \right.$$

$$+ \mathbb{E}\left[Y | S = s_B\right] P\left(S = s_B | T = t\right) \wp_{T | S = s_G}(t) \Big) dt$$

$$= \frac{1}{\ell} \int_{\frac{\eta_U}{2} + \eta_I - \frac{\ell}{2}}^{\frac{\eta_U}{2} + \eta_I + \frac{\ell}{2}} \left(\frac{\mathbb{E}\left[Y | S = s_G\right] P\left(S = s_G\right)}{P\left(S = s_G\right) + 1_{[\eta_U / 2]}(t) P\left(S = s_B\right)} \right.$$

$$+ \frac{\mathbb{E}\left[Y | S = s_B\right] P\left(S = s_B\right) 1_{[\eta_U / 2]}(t)}{P\left(S = s_G\right) + 1_{[\eta_U / 2]}(t) P\left(S = s_B\right)} \right) dt$$

$$= \frac{1}{\ell} \left((\ell - \eta_I) \frac{\mathbb{E}\left[Y | S = s_G\right] P\left(S = s_G\right)}{1} + \eta_I \frac{\mathbb{E}\left[Y | S = s_G\right] P\left(S = s_G\right)}{P\left(S = s_G\right)} \right.$$

$$+ (\ell - \eta_I) \frac{\mathbb{E}\left[Y | S = s_B\right] P\left(S = s_B\right)}{1} \right)$$

$$= \mathbb{E}[Y] + \frac{\eta_I}{\ell} \left(\mathbb{E}\left[Y | S = s_G\right] - \mathbb{E}[Y] \right).$$

Similarly,

$$\mathbb{E}\left[\hat{p} | S = s_G\right] = \mathbb{E}\left[\hat{Y}\right] + \frac{\hat{\eta}_I}{\ell} \left(\mathbb{E}\left[\hat{Y} | S = s_G\right] - \mathbb{E}\left[\hat{Y}\right] \right).$$

As a consequence, the ex-ante utility function of partially aware informed traders is

$$\hat{U}_{PI}\left(\hat{\eta}_I\right) = \left(\mathbb{E}\left[\hat{Y} | S = s_G\right] - \mathbb{E}\left[\hat{Y}\right] \right) P\left(S = s_G\right) - \frac{\hat{\eta}_I}{\ell} \left(\mathbb{E}\left[\hat{Y} | S = s_G\right] - \mathbb{E}\left[\hat{Y}\right] \right) P\left(S = s_G\right) - c.$$

25

Analogously, the ex-ante utility function of fully aware informed traders is, for any given $\hat{\eta}_I$,

$$
\begin{aligned}
U_{FI}(\eta_{FI}) &= (\mathbb{E}[Y|S=s_G] - \mathbb{E}[Y]) P(S=s_G) - \frac{\eta_I}{\ell} (\mathbb{E}[Y|S=s_G] - \mathbb{E}[Y]) P(S=s_G) - c \\
&= (\mathbb{E}[Y|S=s_G] - \mathbb{E}[Y]) P(S=s_G) \\
&\quad - \frac{\hat{\eta}_I \bar{\eta}_P + \eta_{FI}}{\ell} (\mathbb{E}[Y|S=s_G] - \mathbb{E}[Y]) P(S=s_G) - c.
\end{aligned}
$$

Importantly, $\hat{U}_{PI}$ is decreasing in $\hat{\eta}_I$ and U_{FI} is decreasing in both η_I and η_{FI}. The values of $\hat{U}_{PI}$ at the extremes of the domain given by $0 \leqslant \hat{\eta}_I \leqslant \ell$ are

$$
\hat{U}_{PI}(0) = \hat{\gamma} - c, \qquad \hat{U}_{PI}(\ell) = -c,
$$

while, for any given $\hat{\eta}_I$, the values of U_{FI} at the extremes of the domain defined by $0 \leqslant \eta_{FI} \leqslant \min\{\bar{\eta}_F; \ell - \hat{\eta}_I \bar{\eta}_P\}$ are

$$
\begin{aligned}
U_{FI}(0) &= \gamma (\ell - \hat{\eta}_I \bar{\eta}_P)/\ell - c \\
U_{FI}(\bar{\eta}_F) &= \gamma (\ell - \hat{\eta}_I \bar{\eta}_P - \bar{\eta}_F)/\ell - c \\
U_{FI}(\ell - \hat{\eta}_I \bar{\eta}_P) &= -c.
\end{aligned}
$$

Since $c \geqslant \hat{\gamma}$, $\hat{U}_{PI}(0)$ is negative. As $\hat{U}_{PI}$ is decreasing in $\hat{\eta}_I$, it is negative on the whole domain and therefore in equilibrium $\hat{\eta}_I^* = 0$. As a result, $\eta_{PI}^* = \hat{\eta}_I^* \bar{\eta}_P = 0$. In the following, we derive the equilibrium values η_{FI}^* in each of the three cases. Then, η_I^* can be automatically deduced from eq. (1).

1. Given $\hat{\eta}_I^* = 0$, $U_{FI}(\eta_{FI})$ is negative in 0 too, because $c \geqslant \gamma$. Hence, for similar reasons, $\eta_{FI}^* = 0$.

2. Here $U_{FI}(\eta_{FI})$ is positive in 0, given $\hat{\eta}_I^* = 0$. Since $U_{FI}(\ell - \hat{\eta}_I^* \bar{\eta}_P)$ is negative for all $\hat{\eta}_I^*$, this also holds for $\hat{\eta}_I^* = 0$ and, in this case, $U_{FI}(\ell)$ is negative. Moreover, $U_{FI}(\eta_{FI}^*) = 0$ if $\eta_{FI}^* = \ell(1 - c/\gamma)$. Since $c \geqslant (\ell - \bar{\eta}_F)\gamma/\ell$, we have $\eta_{FI}^* \leqslant \bar{\eta}_F$. Therefore, the constraint $0 \leqslant \eta_{FI}^* \leqslant \min\{\bar{\eta}_F; \ell - \hat{\eta}_I^* \bar{\eta}_P\}$ is satisfied, given $\hat{\eta}_I^* = 0$.

3. The reasoning is identical to that in the second point. However, the condition $c < (\ell - \bar{\eta}_F)\gamma/\ell$ implies that U_{FI} takes the value 0 outside the required constraint for η_{FI}. Thus, $U_{FI}(\bar{\eta}_F)$ is positive, given $\hat{\eta}_I^* = 0$. In this case, all fully aware traders get informed: $\eta_{FI}^* = \bar{\eta}_F$.

Proof of Proposition 3

Following the same reasoning of the proof of Proposition 2, since $c < \hat{\gamma}$, we find that $\hat{U}_{PI}(0)$ is positive. Moreover, $\hat{U}_{PI}(\ell) = -c$ is negative and $\hat{U}_{PI}(\hat{\eta}_I^*) = 0$ when $\hat{\eta}_I^* = \ell(1 - c/\hat{\gamma})$.

26

The expression of η_{PI}^* follows from $\eta_{PI}^* = \hat{\eta}_I^* \bar{\eta}_P$. Given such $\hat{\eta}_I^*$, the function U_{FI} takes, for $\eta_{FI} = 0$, the positive value $\gamma(\bar{\eta}_F + \bar{\eta}_P c/\hat{\gamma})$. Furthermore, $U_{FI}(\ell - \hat{\eta}_I \bar{\eta}_P) = -c$ is negative for all $\hat{\eta}_I$ and, given the $\hat{\eta}_I^*$ above, $U_{FI}(\eta_{FI}^*) = 0$ if

$$\eta_{FI}^* = \ell \left(\bar{\eta}_F - c \left(\frac{1}{\gamma} - \frac{\bar{\eta}_P}{\hat{\gamma}} \right) \right).$$

In point 4, the additional condition in the statement ensures that $\eta_{FI}^* \leqslant \bar{\eta}_F$ and so the constraint $0 \leqslant \eta_{FI}^* \leqslant \min\{\bar{\eta}_F; \ell - \hat{\eta}_I^* \bar{\eta}_P\}$ is satisfied, given the $\hat{\eta}_I^*$ above. Then, η_I^* follows from eq. (1).

In point 5, the opposite condition to that of point 4 holds. Hence, the value of η_{FI} which makes U_{FI} equal to zero, given the proper $\hat{\eta}_I^*$, exceeds $\bar{\eta}_F$. As a consequence, $U_{FI}(\bar{\eta}_F)$ is positive, given $\hat{\eta}_I^*$, and so all fully aware traders get informed, i.e. $\eta_{FI}^* = \bar{\eta}_F$.

Proof of Corollary 1

First of all, note that $\hat{\gamma}$ is strictly decreasing in the unawareness level by Proposition 1. Then, consider the statements of Propositions 2 and 3. First, we show that the monotonicity results hold in any equilibrium configuration. In the fifth type of equilibrium, η_I^* is strictly decreasing in the unawareness level, while it is constant in the other equilibria. Similarly, η_{PI}^* is strictly decreasing in the unawareness level in the fourth and fifth equilibrium, and it is constant in the others. On the contrary, η_{FI}^* is strictly increasing in the fourth equilibrium and constant in the others.

We now consider the possibility of moving from one equilibrium to another when the unawareness level rises. We show that the monotonicity results of η_I^*, η_{PI}^* and η_{FI}^* are still valid when this shift occurs.

In the first, second and third equilibria, η_I^*, η_{PI}^* and η_{FI}^* do not depend on the unawareness level. Hence, from each of these equilibria it is impossible to move to another equilibrium type when the unawareness level changes. However, from the fourth and fifth equilibria it is possible to reach other equilibrium configurations. The only possible moves are from the fourth equilibrium to the fifth, from the fourth to the second, and from the fifth to the third. We first discuss why the other shifts are impossible. After that, we show the monotonicity properties of η_I^*, η_{PI}^* and η_{FI}^* across equilibria.

Moving from the fourth equilibrium to the first is not possible because the fourth equilibrium requires $c < \hat{\gamma} < \gamma$ while the first requires $c \geqslant \gamma$. The same reason prevents the shift from the fifth equilibrium to the first.

Moving from the fourth equilibrium to the third is not possible. Indeed, in the fourth equilibrium, $c < \hat{\gamma}$. If the unawareness level increases, $\hat{\gamma}$ strictly decreases and we may have

$c \geqslant \hat{\gamma}$ with the condition in eq. (3) still valid. Nevertheless, the latter is not compatible with the requirement of the third equilibrium: $\bar{\eta}_F < \ell(\gamma - c)/\gamma$. Indeed, since $c \geqslant \hat{\gamma}$,

$$\frac{\ell}{\gamma} \frac{c(\gamma - \hat{\gamma})}{\ell c + (1 - \ell)\hat{\gamma}} \geqslant \frac{\ell}{\gamma} \frac{c(\gamma - c)}{\ell c + (1 - \ell)c} = \frac{\ell}{\gamma}(\gamma - c)$$

and so $\ell(\gamma - c)/\gamma \leqslant \bar{\eta}_F < \ell(\gamma - c)/\gamma$, which is a contradiction.

Moving from the fifth equilibrium to the second is not possible either. Indeed, in the fifth equilibrium, for a given unawareness level, we have $c < \hat{\gamma}$ and the inverse of the relation in eq. (3). Hence, $\bar{\eta}_F < \ell(\gamma - c)/\gamma$ because

$$\bar{\eta}_F < \frac{\ell}{\gamma} \frac{c(\gamma - \hat{\gamma})}{\ell c + (1 - \ell)\hat{\gamma}} < \frac{\ell}{\gamma} \frac{c(\gamma - c)}{\ell c + (1 - \ell)c} = \frac{\ell}{\gamma}(\gamma - c).$$

However, the conditions for the second equilibrium require that $\bar{\eta}_F \geqslant \ell(\gamma - c)/\gamma$ and so they cannot be satisfied.

Moving from the fifth equilibrium to the fourth is not possible. Indeed, if the unawareness level increases, $\hat{\gamma}$ strictly decreases and, in the ratio of eq. (3), the denominator decreases while the numerator increases. As a result, the ratio increases. In the fifth equilibrium the inverse of the relation in eq. (3) holds and it cannot be reversed when the unawareness level rises. Therefore, it is not possible to satisfy the condition for the fourth equilibrium starting from the fifth.

Moving from the fourth equilibrium to the fifth, the monotonicity properties of η_I^*, η_{PI}^* and η_{FI}^* still hold. The shift across these equilibria is possible because, when the unawareness level rises, the ratio in (3) increases and the inequality can get reversed. In the case in which c is still lower than $\hat{\gamma}$, the fifth equilibrium occurs. In this equilibrium, η_{PI}^* keeps the same expression as in the fourth equilibrium, and so it features the same decreasing monotonicity. In addition, η_{FI}^* becomes constant in the unawareness level and its value is higher than all the values achieved in the fourth equilibrium because

$$\bar{\eta}_F > \ell \left(\bar{\eta}_F - c \left(\frac{1}{\gamma} - \frac{\bar{\eta}_P}{\hat{\gamma}} \right) \right).$$

Therefore, the increasing monotonicity of η_{FI}^* is not affected (in the weak sense). Finally, in the fifth equilibrium $\eta_I^* = \eta_{PI}^* + \eta_{FI}^*$ is decreasing in the unawareness level because η_{PI}^* is decreasing and η_{FI}^* is constant. Moreover, the values of η_I^* in the fifth equilibrium are lower than that in the fourth equilibrium, maintaining η_I^* decreasing in the unawareness level when moving from the fourth equilibrium to the fifth. Indeed, by using $\bar{\eta}_P = 1 - \bar{\eta}_F$

28

and the reverse of inequality (3),

$$\bar{\eta}_P \ell \left(1 - \frac{c}{\hat{\gamma}}\right) + \bar{\eta}_F = \ell \left(1 - \frac{c}{\hat{\gamma}}\right) + \frac{\bar{\eta}_F}{\hat{\gamma}} \left(\ell c + (1 - \ell)\hat{\gamma}\right)$$

$$< \ell \left(1 - \frac{c}{\hat{\gamma}}\right) + \frac{\ell}{\hat{\gamma}\gamma} \frac{c(\gamma - \hat{\gamma})}{\ell c + (1 - \ell)\hat{\gamma}} \left(\ell c + (1 - \ell)\hat{\gamma}\right) = \ell \left(1 - \frac{c}{\hat{\gamma}}\right).$$

This ensures that the values of η_I^* in the fifth equilibrium are lower than the value of η_I^* in the fourth equilibrium.

Moving from the fourth equilibrium to the second, the monotonicity properties of η_I^*, η_{PI}^* and η_{FI}^* still hold. The shift across these equilibria is possible because, when the unawareness level rises, $\hat{\gamma}$ strictly decreases and we can have $c \geqslant \hat{\gamma}$. In the case in which $\bar{\eta}_F \geqslant \ell(\gamma - c)/\gamma$, we reach the second equilibrium. In the latter, the equilibrium values are constant in the unawareness level. In particular, $\eta_{PI}^* = 0$ is smaller than any value of η_{PI}^* attained in the fourth equilibrium, in agreement with its decreasing monotonicity. In addition, η_I^* is identical (and constant) in the two equilibria. As for η_{FI}^*, we show that its values in the fourth equilibrium are smaller than its value in the second. Indeed, since $\bar{\eta}_P = 1 - \bar{\eta}_F$ and $c < \hat{\gamma}$ in the fourth equilibrium,

$$\ell \left(\bar{\eta}_F - c \left(\frac{1}{\gamma} - \frac{\bar{\eta}_P}{\hat{\gamma}}\right)\right) = \ell \left(1 - \frac{c}{\gamma}\right) + \ell \bar{\eta}_P \left(-1 + \frac{c}{\hat{\gamma}}\right) < \ell \left(1 - \frac{c}{\gamma}\right).$$

Therefore, the increasing monotonicity of η_{FI}^* is not violated.

Finally, moving from the fifth equilibrium to the third, the monotonicity properties of η_I^*, η_{PI}^* and η_{FI}^* still hold. The shift across these equilibria is possible because, when the unawareness level rises, $\hat{\gamma}$ strictly decreases and we can have $c \geqslant \hat{\gamma}$. In the case in which $\bar{\eta}_F < \ell(\gamma - c)/\gamma$, we reach the third equilibrium. In the latter, $\eta_{PI}^* = 0$ is lower than any value of η_{PI}^* attained in the fifth equilibrium, and so its decreasing monotonicity still holds. In addition, the value of η_{FI}^* is identical (and constant) in both equilibria. Finally, the values of η_I^* in the fifth equilibrium are higher than its value in the third equilibrium because

$$\bar{\eta}_P \ell \left(1 - \frac{c}{\hat{\gamma}}\right) + \bar{\eta}_F > \bar{\eta}_F.$$

As a consequence, η_I^* attains its minimum value in the third equilibrium and this confirms its decreasing monotonicity.

Proof of Corollary 2

In the proof of Proposition 2 we showed that, in equilibrium,

$$\mathbb{E}[p|S = s_G] = \mathbb{E}[Y] + \frac{\eta_I^*}{\ell} \left(\mathbb{E}[Y|S = s_G] - \mathbb{E}[Y]\right).$$

By Corollary 1, η_I^* is decreasing in the unawareness level and so is $\mathbb{E}[p|S = s_G]$. Moreover, since $\mathbb{E}[p] = \mathbb{E}[Y]$ is independent of the unawareness level, as well as $P(S = s_G)$ and $P(S = s_B)$, we deduce that $\mathbb{E}[p|S = s_B]$ is increasing in the unawareness level.

Proof of Corollary 3

To compute $var(p)$, we use the three regions of order flows I_1, I_2, I_3 previously defined (and represented in Figure 1) and we recall that $\mathbb{E}[p] = \mathbb{E}[Y]$. Hence, in equilibrium,

$$var(p) = \mathbb{E}\left[(p - \mathbb{E}[Y])^2\right] = \mathbb{E}\left[\left(\mathbb{E}\left[Y|S = s_B\right] - \mathbb{E}[Y]\right)^2 1_{I_1}(T)\right]$$
$$+ \mathbb{E}\left[0^2 1_{I_2}(T)\right] + \mathbb{E}\left[\left(\mathbb{E}\left[Y|S = s_G\right] - \mathbb{E}[Y]\right)^2 1_{I_3}(T)\right]$$
$$= \left(\mathbb{E}\left[Y|S = s_B\right] - \mathbb{E}[Y]\right)^2 \mathbb{E}\left[1_{I_1}(T)\right] + \left(\mathbb{E}\left[Y|S = s_G\right] - \mathbb{E}[Y]\right)^2 \mathbb{E}\left[1_{I_3}(T)\right].$$

By using eq. (6) for the probability $\wp_T(t)$ of any order flow t, we find

$$\mathbb{E}\left[1_{I_1}(T)\right] = \int_{\frac{\eta_U^*}{2} - \frac{\ell}{2}}^{\frac{\eta_U^*}{2} + \eta_I^* - \frac{\ell}{2}} \wp_T(t)dt = P\left(S = s_B\right)\frac{\eta_I^*}{\ell}$$

and, similarly,

$$\mathbb{E}\left[1_{I_3}(T)\right] = \int_{\frac{\eta_U^*}{2} + \frac{\ell}{2}}^{\frac{\eta_U^*}{2} + \eta_I^* + \frac{\ell}{2}} \wp_T(t)dt = P\left(S = s_G\right)\frac{\eta_I^*}{\ell}.$$

Consequently,

$$var(p) = \frac{\eta_I^*}{\ell}\left(\left(\mathbb{E}\left[Y|S = s_B\right] - \mathbb{E}[Y]\right)^2 P\left(S = s_B\right) + \left(\mathbb{E}\left[Y|S = s_G\right] - \mathbb{E}[Y]\right)^2 P\left(S = s_G\right)\right).$$

By Corollary 1, η_I^* is decreasing in the unawareness level and so is $var(p)$.